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THE TECHNIQUE OF TEACHING SECONDARY-SCHOOL MATHEMATICS

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THE TECHNIQUE OF TEACHING SECONDARY-SCHOOL MATHEMATICS

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PREFACE

This volume is the first part of a series of books on the teaching of mathematics in the secondary schools. The series is the outcome of the author's many years of experience in teaching mathematics to pupils of the junior and senior high school, in training practice students intending to become teachers of mathematics, in supervising the work of a mathematics department, and in conducting special-methods courses offered to college seniors and graduates.

The book has been prepared to serve the following purposes:

1. To help the reader deal intelligently with the problems relating to the technique of teaching mathematics.
2. To outline effective procedures, and devices to be used in teaching mathematics.
3. To discuss the problems relating to classroom management.
4. To show the use of tests as a means of improving instruction.
5. To suggest how pupils may be taught to study.
6. To discuss the general and specific objectives of secondary-school mathematics and the general objectives of secondary education.
7. To acquaint the reader with modern tendencies in the teaching of mathematics and in the selection and organization of the instructional materials.
8. To show the relations of the teacher of mathematics to the administration of the school.
9. To acquaint the reader with the literature relating to the various topics discussed in the book in order that he may ascertain the views of other writers.

The book is intended for the following groups of readers:

1. The prospective teacher, the college student who has decided to become a teacher of mathematics. It is aimed to make him realize the need for studying the problems arising in teaching and to prepare him to meet the problems of classroom teaching and management, by giving him such experiences, with practical situations, as he will have to face as a teacher of mathematics.

2. The experienced teacher who is looking for suggestions to help him improve his instruction. He will derive much help from the typical procedures, concretely illustrated and described, in his endeavor to solve in a satisfactory manner the numerous problems which arise in teaching.

3. The supervisor who wishes to recommend to his teachers such procedures as have been carefully tried out and refined through years of practice, who is in search for materials and topics for discussion in teachers' meetings and who is in need of criteria for judging the effectiveness of methods and procedures.

4. Normal-school instructors and instructors in summer schools, who have in their classes prospective and experienced teachers. They will find in the book not only detailed discussions of effective procedures but also exhaustive bibliographies of articles and books by other writers in which the procedures have been discussed.

This volume does not deal with the teaching of specific topics, processes, and materials, or with the methods of selecting and organizing subject matter. These problems will be discussed in detail in the second volume of the series.

The author is deeply indebted to Professor Charles H. Judd, director of the School of Education of the University of Chicago, for the numerous helpful criticisms given while the manuscript was in preparation; to Professor H. C. Morrison, who, as superintendent of the laboratory schools of the School of Education, directed and assisted the department of mathematics in the development of a teaching technique; to Professor W. C. Reavis who has read and criticized in detail several of the chapters; and to Mr. Charles A. Stone, of the University High School, who has used the manuscript in his special-methods classes.

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E. R. Breslich

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CHAPTER I

THE CLASS-PERIOD ACTIVITIES OF THE TEACHER OF MATHEMATICS

An effective teaching technique is essential to success in teaching.—Thirty years ago a college graduate was considered fully qualified to teach the mathematics of the secondary schools without any special professional preparation for teaching. Understanding, insight, and mathematical power acquired in college courses in mathematics, supposedly developed automatically the power to understand and solve the problems arising in the teaching of the lower schools. It was assumed that the inexperienced teacher merely had to choose as models of good teaching the procedures employed by his best college instructors, and to adapt them to his own pupils as well as he could by the method of trial and error. Even today there are those who look upon the work of the teacher of mathematics as something that most any college graduate can do without professional training. Indeed, it frequently happens that classes in mathematics are assigned to teachers whose academic preparation has been in a subject other than mathematics. All that they are expected to do is to make daily lesson assignments, to spend the class periods listening to the recitations of the pupils, and to determine their grades. In many schools large sums of money are spent for physical equipment because its importance is fully recognized, and much time and effort is spent on the reorganization of the courses and on the construction of the curriculum; but the value of an effective teaching technique is greatly underrated.

Effective teaching requires careful planning for the activities of the class period.—Among the numerous activities of school life, none are more important than those of the class period. One may safely claim that the highest efficiency of our schools cannot be reached until teachers have learned how to make every minute of the class period contribute toward the achievement of results. Success or failure of the pupil depend largely on the extent to which he profits

from his class-period experiences. In the classroom he should acquire information, develop right habits of study, learn to express his thoughts clearly and precisely in oral and written form, and train himself to perform his school work in the most effective manner. If the teacher hopes to accomplish the achievement of such results, he must make every possible effort to eliminate unnecessary waste. He must plan the entire unit's[1] work in every detail. He must provide such activities as discussing, drilling, reviewing, measuring, testing, marking, assigning work, learning lessons, and asking and answering questions.

Without careful planning the class period is likely to be an ineffective mixture of some or all of these activities. This mixture is probably the greatest source of waste of class time observed in our schools. It offers little or no opportunity to the pupil to develop initiative, individual responsibility, concentration of effort, and sustained attention. Furthermore, the teacher must plan to introduce each activity at the proper time and for a purpose so apparent that it is not only definite in his mind but clearly conceived by the pupils.

It is a mistake to assume that only the beginner, the inexperienced teacher, needs to make a careful plan for the class period. No teacher should step before his class without having previously formulated a definite plan of the procedure to be followed.

How to start the class period promptly.—The teacher's plan should eliminate from the class period all routine activities that can be performed before class time. The tasks of distributing and collecting papers, putting the equipment in working order, looking after the physical conditions of the room, placing materials where they will be easily accessible, taking attendance, and many others, can all be delegated to responsible pupils who will be glad to perform them in the few minutes preceding the class period. The teacher is thereby left entirely free to supervise the room during the intermission. In a classroom free from disorder the pupils will relax. Gradually their thoughts will turn to study; and when the bell rings, everybody will be at work.

A properly trained group of pupils begins work promptly with-

[1] A statement of the term "unit" in the sense in which it is used in this book is found in chapter vi.

out waiting for the teacher to get them started. They have been taught to keep themselves employed. They will find unfinished tasks to be completed, or they will review and organize the work of the preceding days. A good device for securing a prompt start is to assign at the end of each class period a definite piece of work to be done at the beginning of the next period. The assignment should be something that everybody will be able to do, preferably a continuation or completion of what each pupil happens to be engaged in when the class is dismissed.

It should be added that at the end of the period all work must stop as promptly as it started in the beginning. If a class is held overtime, it will be deprived of part of the intermission which follows. Time wasted at the beginning cannot be made up at the close of the period. If every moment is utilized, the teacher can afford to set aside a minute or two at the end of the period for collecting materials and other routine work. When this has been done, the class will be ready to be dismissed as soon as the bell rings.

Selecting subject matter for study.—One of the first steps in preparing for the class period is to select and organize the instructional materials to be taught or studied. Pupils will work hard if they are given work they can do and if the tasks to be performed appeal to them as worth doing. Hence, the teacher cannot overlook the importance of carefully studying the needs of the pupils, selecting suitable and interesting materials, not too difficult and not too simple, providing motives for study, and securing local materials to supplement the textbook. This calls for keen judgment and broad understanding on the part of the teacher.

Some teachers follow the adopted text without making any changes. Perhaps that is the wisest plan when a textbook is used for the first time and when the teacher is not yet thoroughly familiar with the aims and purposes of the author. Rearrangement, if it is attempted before the latter are clearly understood, might seriously interfere with, or even destroy, those purposes. But in general the content of textbooks needs to be reorganized, especially when the instructional materials are arranged in strictly topical order and in logical sequence, which is not always the arrangement in which they are best learned, and when they are otherwise deficient in organization.

Furthermore, in addition to the minimum to be assimilated by all pupils, textbooks usually contain subject matter intended for purposes of illustration, motivation, appreciation, insight, clear understanding, and practice. Without careful planning, the teacher might misplace emphasis and spend too much time on relatively unimportant factors rather than on the assimilation of the essential principles which constitute the course. To enable the learner to attain a good grasp of these principles, the subject matter may need to be reorganized in blocks or units that are suitable for teaching purposes (chap. vi). The major function of the textbook will then be to supply the necessary assimilative materials and to give the pupil such experiences as will aid him in acquiring real understanding of the various units. For example, in the study of the laws for multiplying algebraic polynomials, the goal is not only to solve correctly each one of a list of separate problems but to work a sufficient number of problems to acquire a high degree of skill in the process and a full and clear understanding of the general principles that are employed in the multiplication of algebraic expressions. Hence, the teacher must carefully select the exercises which best contribute to a clear conception of the principles to be mastered in order that with each exercise the principles gain in clearness.

What has been said of subject matter applies also to method. Effective habits of study and skill in the use of mathematical methods of thought are not acquired accidentally. Each topic, chapter, and unit must make definite contributions to mathematical methods which finally must become so clear that pupils can readily employ them in new problem situations. If a method is so difficult that pupils cannot learn to use it, it is of little value to them.

Taking an inventory of what the class knows about a course or unit.—In beginning a new course, the teacher should inform himself as far as possible about the mathematical abilities of the pupils and the extent to which they have had previous experiences with the instructional materials to be presented in the course. This might be done in several ways: If the pupils' marks in the previous courses can be secured, the teacher will be in possession of information that reflects the judgment of other teachers. Intelligence tests, subject prognostic tests, reading tests, and mathematical achievement tests supply further valuable information. Thus, in a first-

year algebra class a suitable or especially devised test in arithmetic[1] will determine the extent to which individual pupils need teaching in the arithmetic actually used in algebra. Similarly, classes beginning demonstrative geometry may be tested to determine the geometrical knowledge possessed by the pupils. Information of this type helps the teacher to make teaching more effective, to adapt new subject matter to the level of ability of the class, and to identify early those who are likely to become leaders and those who will probably need special attention during the course.

The pupils also are eager to obtain this information in order that they may realize the extent of their deficiencies and take steps to remove them.

What has been said about beginning a new course frequently applies to a new unit. The teacher must inform himself regarding the pupils' experiential background with the content of the unit and regarding the specific knowledge the class possesses with respect to the unit. That failure to do this may result in useless repetition of knowledge previously acquired, which in turn may be the cause of lack of interest and waste of time, may be seen from the following illustration:

A student practice teacher was observed as she introduced the unit on quadrilaterals to a class in plane geometry. It was readily seen that the lesson had been carefully planned and that the student was presenting the material in a creditable way. However, it soon became obvious that as the student proceeded with the lesson the class was losing interest. The pupils were getting restless. Noticing this, the student ceased speaking and asked: "What seems to be the matter?" One of the pupils answered promptly: "We had all this before." "If that is so," the student replied, "then you may tell me what you know about it." Instantly the situation was entirely changed. There was keen attention and interest as the pupils, one by one, told what they knew about quadrilaterals. Practically everything the student had planned to say in the introduction of the unit was brought out by the pupils. It remained only for the student to summarize what had been said. She had accidentally dis-

[1] W. C. Reavis and E. R. Breslich, *Diagnostic Tests in the Fundamental Operations of Arithmetic and in Problem Solving*, Forms A and B (for Grades VII to XII). (Chicago: the University of Chicago Press, 1927.)

covered an excellent method of exploration which gave her a point of view with which to begin the new unit, but she realized that it would have been better to start the unit with such an exploration.

The method of informal discussion, as described above, usually fails to go far enough. It does not identify clearly those pupils who had former experiences with the new materials to be taught, and it does not determine definitely the amount and character of such experiences. The remedy is easily found in an *inventory test.* The advantages of such a test are apparent. Pupils who show knowledge of certain phases of the unit may be excused from the assigned work when these phases are studied, and devote their time to other tasks. Furthermore, whenever the test shows that all pupils are familiar with certain facts, principles, or processes, no further attention needs to be paid to them during the study of the unit. For a detailed discussion of the method of making inventory tests the reader is referred to chapter vii. It will be sufficient here to present one example of such a test which was given preparatory to the study of "Angle Pairs."

TEST—UNIT 5—ANGLE PAIRS[1]

PART I

Complete the following sentences:

1. The sum of the acute angles of a right triangle equals_____
2. A triangle having two equal angles is_____triangle.
3. The side opposite the right angle in a right triangle is called the _____.
4. The sum of the adjacent angles about a point on one side of a straight line is_____degrees.
5. The sum of all the angles at a point just covering the angular space about the point is_____degrees.
6. In the following statements put a check mark for those that are correct and a cross for those incorrect: (See Fig. 1.)

_____*a* and *b* are adjacent angles.
_____*b* and *c* are supplementary angles.
_____*b* and *e* are opposite angles.
_____*b* and *e* are complementary angles.

[1] Charles A. Stone, "A Systematic Procedure in the Development of a Learning Unit in Mathematics," *School Science and Mathematics,* XXVII (October, 1927), 691–701.

_____c and d are adjacent angles.
_____a and c are adjacent angles.
_____b and c are adjacent angles.

7. Draw an angle adjacent to the angle x. (Fig. 2.)
8. Draw an angle that will be the complement of angle r. (Fig. 3.)
9. Draw an angle that will be the supplement of angle z. (Fig. 4.)

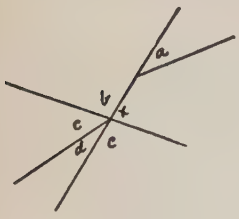

Fig. 1

Fig. 2

Fig. 3

Fig. 4

10. Put check marks in the figures below that contain perpendicular lines. (Fig. 5.)

Fig. 5

11. Draw a figure to represent an isosceles triangle.
12. How many right angles may a triangle have?_____.
13. To what is the sum of the angles of a triangle equal?_____
14. How many perpendiculars can be drawn to a given line at a given point?_____.
15. What is the complement of an angle of 60°?_____.
16. Two angles of a triangle are 60° and 100°. What is the value of the third angle?_____.
17. If the marked angle is 70° (Fig. 6), find the angle X_____.
18. If the marked angle is 50° (Fig. 7), find the value of S_____.
19. In Fig. 8, mark with the letter C, an angle which forms with A, a pair of alternate interior angles.
20. In Fig. 8, mark with the letter D, an angle which makes with A, a pair of corresponding angles.
21. One acute angle of a right triangle is 72°. What is the other acute angle?_____.

Fig. 6

Fig. 7

Fig. 8

Fig. 9

Part II

22. An angle at the vertex of an isosceles triangle is 80°. Find each of the other angles_____.

23. If angle A is twice as great as angle B, find the size of angle C. (Fig. 9.)

24. $X°$ is the supplement of $X°+84°$. Find the angles_____.

25. In Fig. 10, find X and the other angles.

26. In Fig. 11, if $c=e$, prove $a=g$.

27. If $c+f=180°$, prove $f=d$. (Fig. 11.)

28. In Fig. 12, find X and the two angles.

29. Using (Fig. 13), write the equation that would give a value for X. Do not solve.

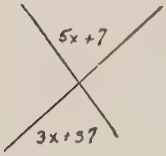

Fig. 10

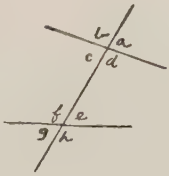

Fig. 11

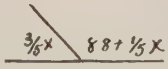

Fig. 12

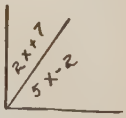

Fig. 13

In discussing the test, Stone says:

In order to determine the experiential background of the pupils with respect to the unit it was decided to administer the test to the class before a study of the unit was attempted. . . . .

Table I shows the responses of the pupils to the test before they have had the opportunity to study the assimilative material of the unit. The table reveals a number of interesting facts. It seems that questions 12, 13, and 16 referring to the interior angles of a triangle might just as well have been omitted. The principle concerning the sum of the interior angles of a triangle had been taught in a previous unit, and was included for review purposes since it is used in developing some of the relations between angle pairs in a triangle. Since the results show that the pupils have mastered

TABLE I

PUPIL'S RESPONSES BEFORE ASSIMILATION COMMENCED

Question	1	2	3	4	5	6							7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	Total Rights
						a	*b*	*c*	*d*	*e*	*f*	*g*																								
Pupil:																																			
A		X			X		X		X	X		X		X	X	X				X	X		X	X		X		X	X		X	X	X			16
B		X	X						X					X	X	X							X					X	X		X	X	X	X		21
C	X	X												X	X		X				X				X	X				X	X	X			23
D	X						X							X	X						X			X	X	X			X	X		X			23
E	X	X	X	X	X		X		X					X	X	X	X		X		X		X	X			X	X	X	X	X	X			13
F	X	X	X	X		X	X		X	X	X	X	X				X	X			X		X	X	X	X			X		X	X	X	X	11
G	X	X		X		X	X			X	X			X	X										X	X			X		X	X		X	18
H	X	X	X	X	X				X	X	X	X	X	X	X		X				X			X	X	X	X		X		X	X	X			10
I					X				X	X		X	X	X	X	X					X		X	X	X			X		X	X	X	X	X	15
J	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X		X				X		X	X	X	X	X		X		X	X	X			8
K		X	X				X		X			X		X			X				X				X			X		X	X	X			21
L	X	X	X				X							X	X						X				X		X		X	X	X	X			21
M	X								X					X						X	X					X					X				28
N		X	X				X							X	X					X	X				X	X		X		X				X	22
O	X	X	X			X	X	X	X		X			X	X	X	X				X			X	X	X	X	X	X		X		X		11
P		X	X		X				X					X	X	X				X	X				X	X	X	X	X	X	X	X	X	X	15
Q					X		X		X					X							X				X	X					X				28
R		X	X				X		X					X	X						X						X		X		X	X	X		23
Totals	8	4	7	13	11	14	6	16	5	12	13	12	14	1	4	12	11	17	17	14	2	18	13	10	5	6	11	7	6	4	6	2	1	11	13	327

this principle any reference to it may be omitted from the unit being considered.

It is further seen that no pupil responded incorrectly to every question, showing that all had some previous knowledge of parts of the unit. Indeed the knowledge of the unit shown by some pupils is surprising. Pupils M and Q gave correct responses to twenty-eight of the thirty-five parts of the test. The effect on these pupils would be deadening if they were required to work through all of the assimilative material. They were called in for a conference and it was revealed that they were familiar with some of the subject matter of the unit having studied the topic of mensuration in the grammar school. These pupils had been doing excellent work in previous units and showed that they were capable. Their I. Q.'s were respectively 123 and 116. They were assigned work that would clear up points in the unit that they did not understand. After completing this work they were excused from the class routine and were permitted to work on projects related to mathematics, and were given to understand that they would be called in at any time for a test on the unit.

Pupils B, C, D, K, L, N, and R showed a good knowledge of the unit, having given from twenty-one to twenty-three correct responses. The above pupils had also studied mensuration in the grammar schools and hence were also familiar with some of the subject matter in the unit. Their I. Q.'s are respectively 106, 135, 97, 119, 101, 117, and 112. Pupil D has a comparatively low I. Q. and does not work as rapidly as the others, but he is persistent and exerts a tremendous effort to do the work. Pupils A, E, L, J, and O made surprisingly low scores in view of the fact that they had been doing excellent work in the past year and seemed to be very alert at all times. Their I. Q.'s are respectively 121, 123, 108, 104, and 115. A conference with the above pupils brought out the facts that pupils A, E, I, and J completed the eight grades of the grammar school. When questioned concerning previous instruction with reference to the particular unit they replied that they thought they had something like it but could not definitely remember. Pupil O completed seven grades of the grammar school and had not had the work of the eighth grade. Thus, her score is not surprising. Pupil J made the lowest score of the group. She was certain that she had not seen any material that had any connection with the unit. Pupils H and B likewise lacked the preparation of the eighth grade, and their scores indicated that they would need special attention during the assimilation of the subject matter. Pupil G with an I. Q. of 116 had been through some of the material in the elementary school and should have scored higher. A study of the work of this pupil revealed, however, that, while he was capable, he would work well only under pressure. It was

evident that he would need careful watching. Pupil F had studied some geometry in the grammar school and gave eleven correct responses. He did well in view of the fact that his I. Q. is 86, and that he is a slow learner. His past work had shown that, while he was very slow in assimilating subject matter, he retained it once he grasped it. Apparently this pupil would need very careful attention.

It is evident that the inventory test offers the teacher an excellent opportunity for making connections between previous information and the new unit and, if necessary, for reviewing briefly related material formerly studied before undertaking the new. With these facts in his possession, the teacher will know when the class is ready to proceed. On the other hand, an informal discussion might be all that is necessary when the unit to be studied is so closely related to a preceding unit that the teacher is already in possession of most of the required information.

Giving the pupil a broad view of the unit.—It is a commonly accepted principle that learning is most economical and lasting if the pupil sees the whole before studying in detail the separate parts. The principle appears to be rarely employed in the teaching of mathematics. Thus, in geometry, theorems are studied one at a time, but the teacher fails to call the pupil's attention to the relation which they have to each other or to the unit as a whole. In algebra special cases and details are presented and studied, but their relations to the unit are not shown. Some teachers establish the relations at the end of the unit by means of reviews and summaries, but only too frequently this is done simply for the purpose of preparing the pupils for the final test. It is, of course, better than no summary or review, but its greatest value is not attained because it comes too late to be thoroughly assimilated. The relations between the various facts and principles which make up a unit should be shown *before* the pupil begins the detailed study of the unit. The teacher should present them to the class by means of clear and concise statements. This should not take much time. The talk should be brief and to the point. Only the larger principles should be sketched without attempt of establishing proof. Care must be taken not to go too much into detail and not to be technical. Since the sketch aims to give the pupil his first view prelimi-

nary to the study of the unit, it may be called a *preview*. It shows in broad outline the road the pupil is going to travel.[1]

The following is an example of a preview[2] introducing the unit on "Areas and Volumes of Solids." The teacher has before him a complete set of models to which he refers freely during his talk, which is about as follows:

The triangles, quadrilaterals, and circles which you have studied earlier in the course are called plane figures because all of their points and lines lie in one and the same plane. The drawings which illustrate the relations and properties of these figures were always made on the page of the textbook, on notebook paper, or on the blackboard. When the points and lines of a figure do not all lie in the same plane, the beginner usually finds it difficult to recognize relationships when they are represented by means of a drawing. Hence before we begin to study figures that are not plane, we shall first make models which actually represent the figures. They will help us to form mental images and to make and understand the drawings used to picture the required relations.

In observing your surroundings you notice many objects of rectangular shape. Bricks, tanks, wagon boxes, freight cars, rooms, houses, and stores are rectangular solids. *Rectangular solids* are the first to be studied in this unit. You will first learn to make models of these solids. Then you will study the surfaces of the models. In this study you will need to review some of the known area formulas. Then the problem of measuring the *content of the interior* enclosed by the surface will be taken up. This gives the *volume* of the solid. Formulas will be worked out for finding the areas of surfaces and the volumes. In doing this it will be necessary to use and review what you know about algebra, and your knowledge of this subject will be extended further.

In the same manner other solids will be studied that are not rectangular. They are the prism, cylinder, pyramid, cone, and sphere.

By means of the formulas developed in this unit, you will be able to solve many interesting problems relating to objects that are of the shapes of rectangular blocks, prisms, pyramids, cylinders, cones, and spheres. Thus, the content of the school swimming tank is found from the formula for finding the volume of a rectangular block. The materials used in building a silo or a cylindrical tank are computed from the formula for finding the surface of a cylinder. The weight of an iron ball will be obtained from the formula for finding the volume of a sphere.

[1] H. C. Morrison, *The Practice of Teaching in the Secondary Schools* (Chicago: University of Chicago Press), pp. 226–27.

[2] E. R. Breslich, *Junior Mathematics*, Book II, pp. 66, 67.

This shows how necessary it will be for you to learn by heart all of the formulas that will be developed.

The first thing you will be asked to do is to make a model of a cube. You may now turn to the textbook and begin to read silently what is said in your textbook in No. 49, p. 68.

Besides giving the pupils a view of the new unit, the preview offers the teacher several other teaching opportunities:

1. It can be used to train pupils to become good listeners. The problem of developing good habits of listening is an important one. A written test following the preview will identify the pupils who have not yet acquired the habit. Special instruction should then be given to them. Further tests will reveal to what extent progress is being made.

2. It makes it easy to show such motives for the study of the unit as will make the unit seem worth while to the pupil. It thereby creates a friendly attitude toward the subject.

3. It presents a broad view of the unit in a manner aiming to arouse the curiosity and interest of the pupils.

These matters are more fully discussed in chapter iii.

Studying the unit.—If the preceding steps have been taken, the teacher has before him a group of pupils whose previous experiences and whose knowledge of the new unit have been ascertained, whose interest in the work which is to follow has been aroused, and who have been shown clearly the road they are going to travel. The class is now prepared to take the next step, i.e., to attain the fullest possible understanding and control of the detailed instructional material of the unit, and to assimilate thoroughly all the important concepts, principles, and facts. This step is commonly known as *assimilative study*. It includes a variety of procedures and activities to be described in the following pages and to be more fully discussed in chapter ii.

One important method of study is to gain from reading the textbook and through the solving of problems the understanding of the unit and of the specific principles outlined in the preview. This is the time for *directed study*. No definite time limit can be set. Study may continue for hours, or even for days, without interruption. The responsibility for assimilating the material rests entirely on the pupil, but the teacher is fully occupied with watching, direct-

ing, and aiding the pupils who need help. Occasionally study is interrupted to listen to some member of the class who gives a report on a section of the textbook. This may be followed by questions or by a brief discussion. Or, if the teacher finds that the same difficulty is encountered by a number of pupils, he may explain the matter to the class as a whole to avoid needless repetition of explanations. However, immediately after each explanation, all pupils should return to the study of the textbook.

In general, when pupils have reached the seventh grade, they have mastered the mechanics of reading, but usually they have not learned how to get the thought from the page of a textbook in mathematics. This calls for training in slow, deliberate and concentrated reading and rereading of sentences and paragraphs; for meditating, and associating the new with the old. This is very different from the type of reading employed in most of the other school subjects, and decidedly different from the superficial and hurried reading of newspapers or stories. Indeed, training in this type of reading is needed virtually throughout the whole period of secondary mathematics. It is intended here merely to call attention to this need. A detailed discussion of methods of developing proper habits of reading is given in chapter iv.

But even when pupils have learned to read mathematical discussions, they are not always able to study a complete unit in mathematics without assistance. Within every unit new processes are to be learned, and new skills are to be developed in which reading must be supplemented by additional teaching and discussion. In such cases the teacher presents the new processes and develops the new principles. Thus, during the period of study the teacher occasionally interrupts class work to teach or to give aid and guidance to the pupils as they try to attain understanding of the processes and principles to be learned. This method of teaching is sometimes called the *developmental method.* As far as possible the pupil is led to *discover* the new facts, although actually the teacher formulates the steps and shows why they have to be taken. Thus, in teaching a class to solve a quadratic equation like $x^2-5x+6=0$ by the factoring method, the teacher not only shows the steps in the process but shows why any solution of $(x-2)\ (x-3)=0$ is also a

solution of $x^2-5x+6=0$, and why the equation $(x-2)\ (x-3)=0$ is satisfied when either factor is zero.

Although this method of teaching is widely used, a word of warning is in order. Class observation shows that only an unusually capable instructor seems to be able to use it successfully. Most teachers commit the error of falling into the lecture method, or waste time by attempting to draw from the pupil, by means of questions, information and knowledge which he does not possess and therefore cannot give. Furthermore, few pupils of the high-school level are able to assimilate new ideas when they are presented orally and rapidly in succession. Not being able to follow a lengthy discussion, they soon lose contact with the speaker, attention wanders, and the lecture ceases to be intelligible.

Evidence of the insufficiency of the lecture method for high-school pupils is not difficult to secure. It is often observed that explanations of problems are of little or no value to those pupils who have difficulty with them, because they are not able to follow the explanation. When the explanation of a pupil is depended upon to teach the other pupils, there are always some who understand little or nothing of what has been said.

An experiment to show the inefficiency of the lecture method is reported by Georges.[1] He finds that even under most favorable conditions the method makes a very poor showing, and comes to the conclusion that the explanation is probably to be found in the fact that most pupils do not develop the habit of good listening unless they receive definite instruction in listening.

Another investigation attempting to determine the relative values of the lecture method and question-answer method of instruction shows that for pupils of Grades VII, VIII, and IX the lecture method is the less effective, while there is little difference in the results for Grades X, XI, and XII;[2] and that it is less effective for pupils in the low fourth of a class, while it gives better results with pupils of the upper fourth. Thus, the lecture method does not seem to be the best method for pupils of the lower grades or for

[1] J. S. Georges, "The Efficiency of the Lecture Method," *Catholic School Journal*, XXVII (January, 1928), No. 8, 353–54.

[2] Grover H. Alderman, "The Lecture Method versus the Question-Answer Method," *School Review*, XXX (March, 1922), 205–9.

those of low ability in any grade, but it becomes more effective as the pupil continues in school work.

As pointed out in the foregoing paragraph, if the lecture method is to be used, it is necessary to give instruction in ways of listening. Such teaching might be followed by tests designated to determine to what extent pupils have acquired the habit. On the other hand, a method by which pupils and teacher together work out new processes has its place in teaching. It is particularly well adapted to giving instruction in ways of attacking and solving problems. This should be an outstanding characteristic and purpose of the method whenever it is used.

Occasionally during the study of a unit in mathematics *assignments* have to be made. This calls for another step in the teaching cycle which is exceedingly important in mathematics. For, many pupils become discouraged when, day after day, they are unable to do the assigned work. To be sure, the pupil who knows how to study might be able to do what is expected without assistance by the teacher, but in the case of the majority of pupils the teacher must proceed very carefully. Before making the assignment, he not only should develop new subject matter and give directions for doing the work but should give a clear "exposition" of the process that has been studied or taught. The various steps in the process are then definitely formulated; and without interruption or explanation, it is shown how they apply in a particular problem. Thus, after teaching the pupil how to solve the quadratic equation by factoring, the teacher selects an example and works out, on the blackboard, the complete process, in the right form and arrangement, as follows:

Given: The equation $x^2-5x+6=0$.

1. Factor the left member. This gives $(x-2)\ (x-3)=0$.
2. Put each factor equal to zero, i.e., let

$$x-2=0 \text{ and } x-3=0.$$

3. Solve each of these equations, and the results are

$$x_1=2 \text{ and } x_2=3.$$

4. Check the results in the original equation.

This step in preparation for making an assignment is called *exposition* or *presentation* of the method or process to be learned. It tells the pupil definitely what he is to do when he begins the assigned work, and the order in which he must proceed. Should he later encounter difficulties, he will be able to make an analysis of the situation and to determine exactly where assistance is needed. He will then be in a position to ask definite questions about the work to be done. This is very different from the usual indefinite "I don't know how to do this work," or "I don't understand any of this." When a pupil has learned to formulate questions, he is usually able to answer his own question, because the making of an analysis of a difficulty on which help is needed is often the first step toward overcoming the difficulty.

Teachers and pupils are apt to make the mistake of rushing to the applications and exercises before the principles or methods which the examples illustrate are completely understood. In their eagerness to finish the assigned work, these pupils begin too soon with the exercises, and then look back in the textbook to learn just enough about the principles to enable them to prepare the "lesson." The procedure may satisfy the lesson-hearing teacher that the lesson has been done, but to the pupil it is nearly as useless as copying another person's work. It must be remembered that not the separate exercises but the assimilation of the general principles, rules, and processes of which the exercises are illustrations or applications are the real objectives of learning.

In the exposition definite instructions as to procedure are given; but even when they are clearly understood, the pupil may be unable to apply them. Some new and unexpected difficulties may arise, and more or less help may be needed by individuals or by the whole class. The pupil at this stage may still fail to make the connection between the general suggestions given by the teacher and the particular problem he is to do. Before making the assignment, the teacher must take another step. He must obtain objective evidence as to the pupil's understanding. This evidence is easily and quickly obtained as follows: The whole class is sent to the board, and the same problem is assigned to all pupils. If they work readily and independently, they are properly prepared for the assignment. If not, the process should be retaught and re-presented until it is fully

understood. Then, and only then, an assignment of similar work should be made. Thus the making of an assignment should be preceded by the following steps:

1. The pupils study the new method, principle, or process to be mastered. This may be done by letting them read the discussion in the text or by having teacher and pupils develop it together. Ample opportunity is to be offered for questions.

2. A clear exposition is now given by the teacher, in which the various steps in the complete process are briefly exhibited in the right form and best arrangement, and in which, without further interruption or explanation, it is shown how they apply in a particular problem or exercise. Further opportunity is then offered for questions.

3. Objective evidence is obtained as to the pupils' understanding. This is easily done by sending the class to the board and assigning a problem to all pupils. If they work readily, they are prepared for an assignment. If not, all work must stop and further teaching must be done.

4. When it is apparent that the pupils are able to do the work, a supervised study assignment is made. This gives the teacher another opportunity to aid those individuals who need further help.

Frequently, the assigned work will be finished during the class period, but sometimes only a part will be done. The remainder may then be completed in the pupil's study period or at home. To take care of individual differences, additional home work may be assigned to the more capable pupils.

The importance of taking the foregoing four steps in making assignments cannot be overemphasized. For one of the major difficulties with home work is that the pupil fails to make the connection between the assignment and the particular problems or exercises he is to work.

The practice of making assignments in mathematics at the beginning or at the end of the class period is questionable, even when the teacher gives suggestions as to the way the work is to be done. Definite connections must be established between class work and the work to be done at home. When the teacher does this in school, objections to home work will cease.

Certain types of home work are as common as they are objec-

tionable. Teachers sometimes seem to expect the impossible of the pupil. To cover the ground of the course, they often assign work that many pupils are unable to do because the teacher failed to realize the difficulties which pupils would experience in trying to master that particular assignment. The conscientious pupil, who does not rest until satisfied that every detail is completed, who does all and more than is expected, who really is not in need of this heavy work, is overburdened. On the other hand, the superficial, easy-going pupil is soon done with the task. He idles away his time. He does not worry when his work is poorly prepared or incomplete and soon gives up trying. He finds it more convenient to secure help from his classmates or elsewhere.

When a class covers ground at a rate too fast for the slow pupil, who cannot do as much as his more gifted classmates, home work does not help him to catch up. Nor will it be of any use to admonish the slow pupil to study harder and to put more time on home work. That type of pupil generally works too hard. What he needs is to learn how to study. Additional assignments only increase his burden, cause more waste of his time, and add to his discouragement.

When the pupil's progress in a course depends largely on his understanding of work done outside of the classroom, much of the time of the class period must be spent in obtaining evidence to show what has been learned. There is much repetition, discussion, and wandering from the subject. Individual difficulties, of no particular interest to many, are being explained by the teacher or by pupils. The whole procedure is wasteful as to time and exceedingly questionable.

If the class period is used entirely for teaching and for individual class work, every minute is made to count toward helping pupils acquire knowledge and toward developing correct habits of study. The teaching situation is greatly clarified. The teacher observes, and therefore understands, the difficulties encountered by the pupils. He knows to what extent every member of the class makes an effort, and succeeds; and he is not deceived as to the true ability of his pupils.

Furthermore, when the pupil does the most essential work in the classroom, it should not be inferred that he is to be idle outside of school. When every phase of the work has been explained in

class and is understood, it still requires a special effort on the part of the pupil to make it his own possession to retain it permanently. He must review and reorganize what he has learned in the class period. This type of work is exceedingly valuable as home work. When pupils are taught effective methods of study in school, individual interests will be developed. Reading may be suggested which will stimulate further these special interests. Such voluntary home work is highly desirable and of greatest educational value. It will create a new attitude toward school work. Pupils will work, not because an assigned task is required by the teacher, but because they have a desire to learn and find pleasure in it. New and higher standards are being set up when pupils work primarily for themselves and not for the teacher.

Practice work.—In the anxiety of completing the unit, teachers frequently make the mistake of not furnishing enough practice during the study of the unit. In mathematics such practice is a necessary experience in the learning process. The exercises usually given in textbooks are assimilative materials and contribute to the understanding of the process or principle to be learned. After they have been worked, further practice may or may not be needed. If given, it must be planned for as carefully as all other activities. Its purpose is to help the individual retain permanently the important facts, and in some cases to make certain processes automatic. It engages the whole class but aims to reach the individual. It provides for the needs of all pupils. Different types of practice work should be employed for the various teaching situations.

At this time a word of warning is in order regarding drill in secondary-school mathematics. Teachers are apt to feel that it is as essential in algebra as in arithmetic to keep up mechanical drill until the pupils are able to make automatic responses. In secondary-school mathematics there is danger of overdoing drill and carrying it too far. Comparatively few situations arise in which pupils are called upon to work at a high rate of speed. On the other hand, accuracy is far more essential than speed, and it is important at all times that pupils should work deliberately and carefully. Rapid workers are frequently very inaccurate because they hurry over any obstacle that offers a difficulty without attempting to overcome it. It is better advice in general to urge pupils to take time

than to hurry, except in the case of a few processes and skills that are used most often and are fundamental to later work. As a rule, understanding is to be attained not by mechanical repetition of a process but by means of frequent experiences in which the process occurs. However, such principles as the laws of addition and subtraction, of signs, and of exponents, should be so thoroughly mastered as to require but little of the pupil's attention when they occur in problem situations. Mechanical repetition will aid in attaining such familiarity. Many of the newer textbooks contain sufficient practice material to satisfy all needs for practice.[1] Otherwise supplementary practice material will have to be supplied by the teacher.

Drill should not be started too early, and it should not be confused with teaching. If drill does not work well, this may indicate that the class is not ready for it, in which case it should be stopped and more teaching done. To substitute drill for teaching is a misconception and one of the most disastrous errors in the teaching of high-school mathematics. Teaching and understanding are first. When a process is not clearly understood, drill might do considerable harm because it might establish and perpetuate undesirable habits of work which later will have to be broken down.

Practice work cannot be successful unless the pupil sees the advantage and importance of it, and is interested. Competitive oral and written timed practice exercises, repeated practice for the purpose of surpassing former records, standardized practice material for the purpose of comparing records attained with given norms, are all very effective. When interest is lacking, practice turns into drudgery, an antagonistic attitude toward it is developed, and it becomes useless. For this reason, it must never be kept up until the point of fatigue is reached.

Reviewing and organizing.—Reviews are often overdone or misdirected. Some teachers even begin new courses with an extended repetition of the work done previously. Generally this is equivalent to going over the old work in the old way. Its real purpose is to correct deficiencies caused by inefficient teaching in the preceding courses, but it is ineffective because pupils see no immediate need for it and have little or no interest in any deferred values which the

[1] See E. R. Breslich, *Senior Mathematics*, Book I.

teacher may set up. Reviews under such conditions waste time and are of little value.

Pupils come to a new course expecting to learn something new and different, something which promises to be more interesting and valuable than what they had before. A formal repetition of previous work is disappointing. It destroys all the interest and enthusiasm which the pupils bring to the course.

Another ineffective type of review is too frequent mechanical repetition of assimilative materials during the course, as is the case when, in geometry, pupils are sent to the board day after day to write the same proofs over and over and to recite them to the class, and when, in algebra, the same problems are assigned again and again to be worked out on the blackboard and to be explained to the class. Many pupils are thereby forced to listen to explanations that offer nothing new. They are marking time; and in some cases, when they discover that all of the facts are going to be explained a number of times, they gradually develop the habit of deferring all serious study to a later time.

Reviews are necessary, but they should always contain a new element or give a new view of the old. They should be given during the course when and where need for them arises. When a pupil finds that he is ignorant of, or weak in, a certain type of work, he will be anxious to review it. For example, pupils are not interested at the beginning of a course in geometry in reviewing the process of finding square roots; but if, later in the solution of a geometric problem, the extraction of the square root of 2 is required, a real motive for review is offered to those who do not remember the process. All then realize the importance of reviewing, because they see that they cannot proceed until they understand the process. If in a geometric construction a need arises for another construction taught previously, a good opportunity presents itself for a review. When old theorems are employed in new proofs, the methods of proof of the old theorem may be recalled. When a new principle is established, other related principles previously studied in the course may be reviewed. To the pupil such reviews should appear to be incidental, but the resourceful teacher will plan for them carefully to bring them in systematically, at the right time and in the right place.

To some pupils a review assignment means little else than that there is nothing to do for the next day. On the other hand, if pupils are taught to form the habit of setting aside a few minutes every day for reviewing and organizing all they are expected to know about the unit, they will become thoroughly familiar with all the important facts and processes of the whole unit, and when it is completed there will be no need for a formal review at the end.

The method of reviewing systematically throughout the unit has been found to be very effective. It applies particularly to the arithmetic of secondary-school mathematics. Algebra and geometry offer unlimited opportunities for keeping under constant review all the fundamental processes of arithmetic.

In geometry the proof of a theorem is sometimes so similar to that of another previously proved that it may be used as a review and test of understanding. For example, if the pupil really understands the proof of the theorem: "The bisector of an *interior* angle of a triangle divides the side opposite the angle into two parts whose ratio is equal to the ratio of the other two sides," he should have no difficulty in proving the corresponding theorem for the bisector of the *exterior* angle. If he has proved the theorem expressing the square of the side opposite an *acute* angle of a triangle in terms of the other two sides, he should be able to work out the proof for the side opposite an *obtuse* angle without any particular difficulty.

Pupils in general do not know how to review. If they have never received definite instructions in ways of reviewing, many will form the notion that reviewing is identically the same as cramming.

For the purpose of good teaching and effective learning, instructional materials in the better textbooks are arranged in psychological order. It is therefore necessary, if the pupil is to understand and appreciate fully the logical connections of the subject matter, that he reorganize it logically so as to make the important principles stand out clearly. He must form the habit of organizing[1] daily all of the work from the beginning of the unit. He must constantly look back over the road he has traveled, each day obtaining a broader view of the unit. If he organizes daily, he will reach the end of the unit with a complete, clear, and logical organization in

[1] See chap. iv.

mind. He will be able to submit objective evidence of this fact by writing a satisfactory organization without referring to his textbook. Furthermore, when attention is given to this phase of the work day after day, the organization is likely to become the permanent possession of the learner. If, however, the pupil neglects to organize his work, a hasty organization at the end of the unit will be of doubtful value. Most likely it will amount to little more than cramming, and its principal value will be lost to the learner.

It is important that the final organization be written in the classroom. During this class exercise the textbook should be kept closed, or the pupil will naturally follow the arrangement of the book, which is psychological rather than logical.

Testing.—In modern instruction testing is an integral part of the teaching and learning process. To most of the activities of the class period the teacher must apply tests to secure objective evidence as to the extent to which the pupil has succeeded. Thus, a test may be given after the preview by asking the pupils to write the preview in their own words. To those who fail in the test a second preview may then be given, after which they are tested again. Likewise, following the exposition of a process, the whole class, or a portion of the class, may be sent to the blackboard and a problem assigned to all. If this test shows that the work is understood, they are ready for an assignment. If not, further teaching, additional directions, or modifications are necessary. To establish whether a class has assimilated the material in its fullest sense, oral tests consisting of numerous rapid-fire questions may be given during, or at the close of, the study period. At the end of the unit a more searching oral quiz is in order. A very effective device is to write problems on small cards and to distribute them among the pupils, who are to work them out on the blackboard and to explain them later to the class. After each explanation opportunity should then be offered to all pupils to criticize and to ask questions.

If, judging from all the evidence obtained, the teacher feels that the unit is thoroughly understood, then, and only then, the final written test should be given. It is common practice in testing to ask questions on assimilative material only. It is highly desirable to include, also, questions which test comprehension of the learning principles and of the pupil's power to apply these principles. Since

grasp of these principles is more important than mastery of detail, tests should be constructed for the purpose of measuring understanding and power.

If the test shows that some pupils have failed to attain real understanding of important facts and principles, further teaching becomes necessary. A second test should then be given. The pupils who are not successful in the second test must be retaught and must continue to study until they can write a satisfactory paper. Although a pupil's work on a unit does not stop until all the essentials are thoroughly understood, there is no reason why he should not be allowed to go on with the class to the new unit while he is being retested on the previous unit.

The subject of testing is further discussed in chapter vii.

The recitation.—To many teachers the recitation is simply the hearing of a lesson previously assigned—a device for finding out whether the pupil has prepared the lesson, and for giving marks. If a lesson is found to be poorly prepared, it is assigned again; and sometimes a difficult lesson is assigned more than twice. When the pupil recites, he is frequently interrupted by criticisms, directions, questions, and challenges by the teacher; and the recitation becomes a mixture of reporting, discussing, teaching, testing, measuring, and marking. It ceases to be an exercise which trains the pupil in presenting in the form of a continuous discourse materials that have been assimilated. The result is waste of class time; and the real object of having the pupil recite, i.e., to give him the opportunity to express his understandings and to show what he really knows, is not attained. The fact should not be overlooked that the real purpose of the recitation is not teaching or testing, but giving the pupil a chance to tell in his own way what he knows. The oral recitation should really be a floor talk, in which the pupil does the teaching. If the work of the unit has been well done, if the pupil has kept up a systematic daily review, he should have little difficulty to express himself. It is evident that before he can recite, he must know something to talk about. The recitation should therefore come at the end of the unit. It follows the organization, and it comes after the final test has been given.

The recitation on a unit in mathematics may be conducted as follows: The various topics discussed in the unit are written on

separate cards. Frequently topics from the pupils' organizations may be re-worded to make suitable topics for the talks. At the beginning of the class period the topics are distributed to the pupils with the request that each, within several minutes to be allowed for that purpose, prepare a discussion of his topic. Every pupil will try his best if he knows he will have to give a good account of himself before the teacher and his classmates. The following are illustrations of topics: "How Angles Are Measured"; "What We Have Learned about the Angles of a Triangle"; "What We Have Learned about Solving Equations," etc.

While the pupils are preparing to recite, all textbooks should be closed. After a few minutes the pupils are called upon, one by one, to give their talks before the class. Standing in front of the room, each addresses his classmates in a forceful and convincing way, in clear and concise statements and well-constructed sentences. He has complete charge of the classroom. He makes free use of the blackboard, models, and other helpful devices, very much as the teacher does when he presents the preview of the unit to the class. The teacher refrains from interrupting, making only the most necessary comments. At the end of the recitation the speaker is subjected to cross-examination by his classmates and to criticisms by the teacher. When all floor talks have been given, the story of the unit will have been completely told. When there are not enough topics for all of the pupils in the class, the recitation may be written. This is then followed by oral recitations, for which only some of the pupils are selected.

Summary.—The suggestions given in chapter i may be briefly stated as follows:

1. As far as possible matters of routine should be taken care of outside of the class period. The class time should be devoted entirely to studying and teaching.

2. A definite teaching procedure should be followed providing for exploration, preview, teaching, making assignments, supervised study, practice, reviewing, organizing, testing, and reciting. There should be a constant effort to develop effective methods of study.

3. Waste of time should be avoided wherever possible. Thus, the exploration eliminates the loss of time caused by teaching what is already known to the pupil. The assignment is made only after objective evidence has shown that the pupils are prepared to do the

required work. Supervised study assists the slow pupil to keep up with the class and teaches the bright pupil to be independent and to keep himself profitably employed at all times.

4. Oral and written practice should not be started too early. Drill is not to be confused with teaching. It should be such that all pupils are kept interested and active, and it should not be carried to the point of fatigue.

5. Pupils must be taught to review and organize their work daily. If postponed to the end of the unit, reviews amount to little more than cramming for the final test and are therefore of little educational value.

6. Tests should give objective evidence of understanding and power as well as of information. If the results of a test are not satisfactory, more teaching should be done and a second test given. The process of retesting and reteaching should continue until it is shown that the pupil thoroughly understands those facts and principles which are essential for further successful progress in the course.

7. The recitation, which is ordinarily the greatest source of loss of time, should come after evidence has shown that the pupils are sufficiently prepared to recite. One of its major aims is to give pupils the opportunity to tell what they know about the unit, and to give practice in oral expression.

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CHAPTER II

EFFECTIVE PROCEDURES AND DEVICES IN TEACHING MATHEMATICS

The teacher of mathematics must master several teaching techniques.—The teacher's success depends to a very large extent on his ability to employ devices and procedures in the presentation of the subject that will make the instructional materials the permanent possession of the learner. The whole of the foregoing chapter was devoted to the detailed steps of a technique which has been found to yield excellent results. The reader will recall definite references to reviews, practice work, directed study, talks by the teacher, use of the textbook, and a number of other procedures and devices. Some of them are so well known and have been so widely adapted and successfully used by teachers that they have become known as methods, or modes, of teaching. Thus, we speak of the lecture method, the drill method, the genetic method, the question-answer method, the heuristic method, the textbook method, the recitation method, the laboratory method, the supervised-study method, the project method, and the blackboard method. Each denotes the more or less exclusive use of a teaching procedure which, under suitable conditions, leads to superior results.

Attempts have been made to determine by scientific experiments which of the various procedures might be the most effective. This has turned out to be a rather difficult problem because the success of a method depends on a number of external factors, such as the age of the pupil, the former preparation of the class, the type of subject matter, and the available physical equipment. Even when these factors are held constant, the choice of method is influenced by the aims to be accomplished. Thus, a method which shows superiority in passing on information might fail utterly when used to develop certain skills and understandings. Moreover, not all teachers are equally successful with the same method even if experiments show it to be the best. Personality and academic or professional preparation will affect the outcome in no small degree.

It is evident that the teacher's choice of procedure should not be determined alone on the basis of results established by scientific experiment. Furthermore, even the best method should not be used to the exclusion of all others. The teacher who attempts it runs the risk of making class work mechanical and monotonous. Variety is essential to successful teaching. The best plan seems to be to make a careful study of each technique, to analyze it, and to master its detailed steps. There are good and bad features in all of them. After eliminating the undesirable features of all, let the teacher learn to become an expert in several methods. He will then have to use his best judgment as to when and where a particular procedure is to be employed. Combinations will most likely secure better results than the exclusive use of one method.

Success of a method of teaching is easily judged in terms of the progress a class makes within defined time limits, the extent to which pupils are interested and respond or are actively engaged, and the understandings attained. Economy of effort and time, pupil response, and attainment of results are the principal characteristics of an effective teaching procedure.

Instruction must aim to reach the individual.—Originally, all instruction in schools was individual. The teacher, sitting at his desk, heard one pupil at a time recite a previously assigned lesson and gave criticisms, suggestions, and instruction when and where they were needed. The other pupils of the class were seated in various parts of the room, busy with the study of their lessons. As education grew in popularity and as the number of pupils increased, it became evident that those studying the same lesson could be instructed simultaneously, and individual instruction gave way to mass instruction. Under this plan it is still aimed to reach the individual, although instruction is given to the group. Mass instruction cannot be as effective as individual teaching, because it is necessarily directed to the "average" pupil, and the special needs and capacities of the individual cannot be sufficiently taken into consideration. The bright pupil usually has an easy time, while the slow is expected to do additional work. Extra work to be done outside of class is largely depended upon to provide for individual differences.

Any good teaching procedure must reach pupils individually

although it may engage the group as a whole. Some procedures, however, attempt to facilitate individual progress by discarding class instruction entirely. Usually a set of assignments is made which the pupils are encouraged to work out at their individual rates, and on which they are tested individually. Group discussions are then limited only to such matters as are of interest and assistance to several pupils.

Both group and individual instruction have certain advantages and disadvantages, and the extreme use of either method is therefore of doubtful value. It will be shown that there are procedures which combine the two methods and thereby secure the advantages of both.

The lecture method.—Lecturing to high-school pupils, as it is usually observed in high-school teaching, is a modified form of college instruction. High-school teachers of mathematics do not, as a rule, present their subject in the form of a series of lectures on which the pupils take notes which they are to study later outside of the classroom. Nevertheless, many teachers have the habit of lecturing, or talking, for a long time when they present to their classes the work to be learned. To be sure, in certain situations talks are necessary. It will be remembered that the "preview" of a unit is a carefully planned talk in which the teacher presents to the class in broad outline what is to be studied later in detail. Such a talk is a most convenient device of passing on information to a group as a whole. If followed by a test, it is also a very effective device for training pupils to listen carefully to what is being said. It keeps their minds active summarizing, associating new ideas and gathering precise information. Likewise, the "exposition," which follows the teaching of a new process, is a brief, concise, uninterrupted talk by the teacher. Here the pupil's mind follows attentively because he knows that he will be called upon immediately to apply what is being said.

The lecture method fails when the teacher talks aimlessly, repeating unnecessarily sentences or parts, or drawing in materials unrelated to the matter under discussion. Such a talk often amounts to little more than light entertainment which requires but little effort of thought on the part of the pupil. To be effective a talk must be clear and brief. When it is too long the pupil's mind begins

to wander. Points are not understood, and he ceases to be able to follow. The talk now goes over his head. Ideas are presented too rapidly to allow time for thinking and assimilation, and the pupil's attitude becomes passive and indifferent. He decides to write down as much as he can and to do his thinking and studying later outside of the classroom. Gradually he develops the habit of postponing all serious study to some more convenient time. Lectures, or talks, by teachers will accomplish results only if they are carefully planned, delivered, and followed by some kind of check or test. Otherwise, they are wasteful of class time, and no real benefit is derived.

The question-answer method.—The discussion of the preceding methods has shown the importance of questioning as a factor in the teaching and learning processes. The well-formulated question stimulates the learner to think and to recall. It tests his information. It is a challenge to his best effort and arouses interest. The question is usually addressed to the class as a whole, but only one pupil is called upon to reply. Sufficient time must therefore be allowed the pupil to find the answer. The method is not to be confused with drill where a series of rapid-fire questions is sometimes used with good success.

As valuable as the question-answer method may be for such purposes as stimulating thought and testing the pupils' knowledge, there are dangers in its exclusive use. When the questions are too difficult, the pupils are easily discouraged. We cannot draw from the learner information which he does not possess. It is not easy to distribute the questions so as to give all pupils equal opportunities. When questions are so easy that they require only "yes" and "no" for an answer, they do not stimulate thought. When teachers simply read the questions from the textbook, the work becomes monotonous and formal. If the pupil is hurried and not given enough time to think, he cannot do justice to himself. Unless pupils are given the time needed to formulate their answers, they form the habit of careless oral expression. Often they do not think at all but try to say what they think the teacher expects them to say. Thus, one of the most valuable teaching devices may fail unless the teacher who uses it has made a real study of the possibilities of the art of questioning and has acquired the skill to employ it effectively.

The genetic method.—This procedure is especially well adapted

to the development of new subject matter. The teacher guides the class as a group—giving information, teaching, supplementing the textbook, leading the pupils toward discoveries, and teaching them how to attack and solve problems. The pupils contribute freely. They make suggestions and accept or reject each other's proposals. They are free to ask questions and to state their own individual difficulties. They study and are taught at the same time.

The procedure is widely used but it is difficult, and therefore successful only in the hands of a skilful teacher. Most teachers who use it make the mistake of rambling and asking aimless questions. If used correctly, the method gives the teacher his best opportunity to arouse interest, to develop appreciation and pleasure, and to teach methods of attacking and solving problems.

The heuristic method.—Like the genetic procedure, the heuristic method aims to correct the error of merely passing on information as a substitute for teaching. It differs from the genetic in the sense that it is an individual method. It tries to reach the individual pupil and to change him from one who simply agrees and accepts knowledge into one who thinks independently and is mentally active. By means of a series of carefully worded questions the teacher leads the pupil to understanding and discovery. Like most individual methods, it retards the progress of a class at the beginning, but the advocates of the method claim that whatever loss of time may be caused will be more than regained later on account of the pupil's growth in power and skill. The procedure is difficult to administer because the teacher must constantly adapt his work to the mental abilities of the various members of the class, and must keep all pupils interested while he is working with one of them. The most serious mistake is made when the teacher tries to lead a pupil to accomplish work which is altogether too difficult for him. Invariably the class loses interest and becomes inattentive, which always results in waste of time.

The laboratory method.—Like some of the other procedures, the laboratory method grew out of the attempt to eliminate the practice of simply "passing on" information. As the name indicates, the pupil is to do his work in the schoolroom rather than at home. In that sense it is the opposite of the traditional home-work–recitation plan. Furthermore, experiments are set up which the pupil is

to perform in the mathematical laboratory. The method stresses the applications of mathematics. The activities are those of measuring, counting, estimating approximately, cutting figures, folding paper, weighing, observing, generalizing, and discovering. The pupil learns by doing. He may work alone or with a small group. He derives the abstract facts and laws from concrete experiences. The method is excellent for developing the meanings of new concepts and principles.

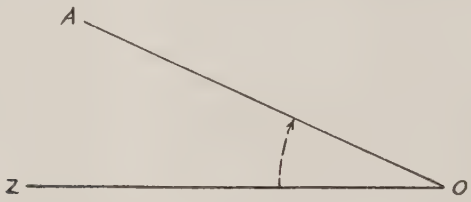
The need of suitable equipment for carrying on laboratory work is evident. The method requires good desks or tables, a library, models, drawing instruments, calipers, thermometers, levers, balances, scissors, and many other materials. Lack of equipment has kept many teachers from giving the method a trial.

Those who object to the use of the laboratory method point out that its exclusive use retards progress; that the pupil does not have enough contact with the instructor; that he does not always make independent discoveries, because he has other pupils help him in his work; and that the lack of group discussions which the method implies is a loss to the learner.

Nevertheless, the laboratory procedure may be adapted successfully to certain types of work which require but a limited equipment. It may be used to supplement rather than to replace class work. The following examples illustrate the procedure.

Example 1:[1] *Finding the sum of the angles of a triangle.*

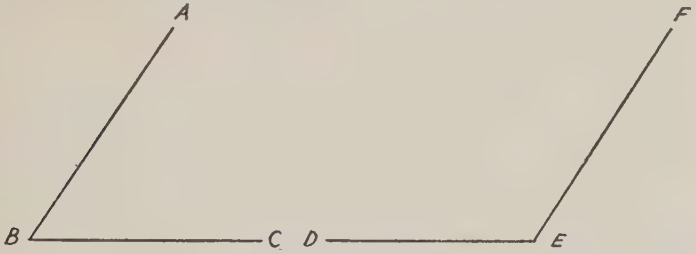
1. Estimate the size of $\angle ZOA$ below. Then measure it.



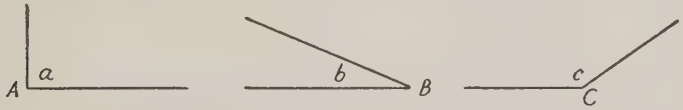
Suggestion: Place the zero-line of the protractor along OZ. From the left-hand zero-mark pass along the rim to the point where OA, or the extension of OA, intersects it. Write the result near the vertex inside the angle. To check your result, measure the angle again by placing the zero-line along OA.

[1] E. R. Breslich, *Seventh-Year Mathematics*, pp. 94, 95.

2. Estimate the sizes of $\measuredangle ABC$ and DEF below. Then measure them. Check your results, as in Exercise 1.



3. Estimate the sizes of the angles below and check your results by measuring them.



	Estimated	Measured	Error	Kind of Angle
A.....				
B.....				
C.....				

4. State the values of a, b, and c.

5. Draw angles of various sizes. Estimate the size of each and then measure them. Continue this until you can do it quickly and accurately.

6. Draw a large triangle and measure the angles. What is the sum of the angles?

7. Draw several triangles and measure the angles. What is the sum of the angles of each triangle?

Example 2:[1] *Finding the circumference of a circle.*—With a tape line, measure carefully the distance around a circular jar or tin can. Make a table as shown below and write the results in the second column.

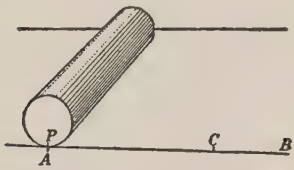
Objects	Distance Around	Diameter	Distance ÷ Diameter
Glass jar.....			
Baking-powder can.....			
Circular plate.....			
Pipe.....			
Glass.....			
Half-dollar.....			
Circle.....			

[1] *Ibid.*, pp. 132, 133.

Measure the distance around other circular objects, such as plates or phonograph records, and place the results in the table.

Cut a circular piece of cardboard and measure the distance around the circle. Enter the result in the table.

The curved surface of the **cylindrical** object below is bounded by circles. On one of the circles mark a point P. Place P on point A of a straight line AB and roll the object, without sliding, so that the circle always touches the line, until point P touches line AB a second time, at C.



The length of the circle may then be found by measuring AC. Write the result in the table.

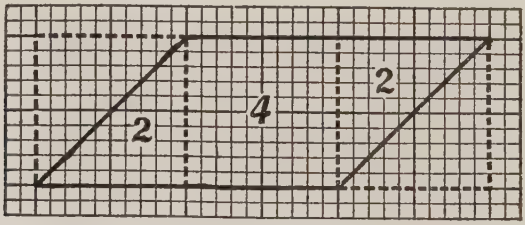
Measure the diameters of the circles you have measured and write the lengths in the table.

For each object find the ratio of the length of the circle to the diameter. With careful measurements these ratios should all be the same.

Find the average of the ratios in the table.

The length of a circle, you remember, is called the "circumference." *For all circles the ratio of the circumference to the diameter is the same.* The ratio, computed approximately to three figures, is 3.14, or $3\frac{1}{7}$. The exact value of the ratio is denoted by the Greek letter π (read *pi*). It stands for the letter *p* and is the first letter in the Greek word for *perimeter*.

Example 3:[1] *How to find the area of a parallelogram.*—To find the area of a parallelogram, proceed as you did with the rectangle. Place the parallelogram upon squared paper and count the number of square units contained in it. In most cases this process would be neither so convenient nor



so practical as that of finding the area by means of a formula. A formula can be easily worked out for any parallelogram as follows:

Draw a large parallelogram $ABCD$ on a stiff piece of paper.

From B draw BC' perpendicular to the side DC.

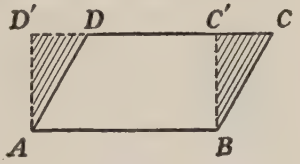
Cut the parallelogram from the paper and then cut off triangle BCC'.

[1] *Ibid.*, pp. 224, 225.

Move the triangle to the position ADD'.

What kind of quadrilateral is $ABC'D'$? How does the area of $ABCD$ compare with that of $ABC'D'$? How would you find the area of $ABC'D'$?

Since the area of the rectangle is $(AB)\cdot(BC')$, the area of the parallelogram is also $(AB)\cdot(BC')$.



Calling AB the *base* of the parallelogram and BC' the height, or *altitude*, we may state the result in this way:

The area of a parallelogram is equal to the product of the base by the altitude. Translating this statement into symbols, we have the formula $A=bh$.

Literature relating to the laboratory method.—The reader will find further suggestions on the laboratory method in the following references.

Austin, C. E. "The Laboratory Method in Teaching of Geometry," *Mathematics Teacher*, XX (May, 1927), 286–93.

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Fawdry, R. C. "Reformed Mathematical Teaching," *Mathematics Teacher*, XVII (February, 1924), 65–71.

Harper, G. A. "Experimental Geometry," *ibid.*, XV (March, 1922), 157–64.

Hefferman, J. K. "The Introduction to the Study of Geometry," *ibid.*, XIX (February, 1926), 78–80.

Hornbrook, A. R. *Laboratory Method of Teaching Mathematics.* New York: American Book Co., 1895.

Nyberg, J. A. "Geometry a Laboratory Science," *School Science and Mathematics*, XXIV (December, 1924), 948–53.

Pratt, G. V. "Popularizing Plane and Solid Geometry," *Mathematics Teacher*, XXI (November, 1928), 412–21.

Pyle, John O. "Geometry a Laboratory Science," *School Science and Mathematics*, XXIV (December, 1924), 956–57.

Shippy, V. Z. "An Example of Geometry Teaching by the Laboratory Method," *Mathematics Teacher*, XVII (May, 1924), 287–89.

Stone, Charles A. "A Laboratory Method of Teaching Mathematics in the Classroom," *ibid.*, XVII (April, 1924), 209–23.

Sutton, F. H. "Geometry a Laboratory Science," *School Science and Mathematics*, XXIV (December, 1924), 954–56.

Directed study in mathematics.—By directed, or supervised, study is meant studying in the classroom under the direction of the teacher, under physical and psychological conditions most favorable for study, the pupil acquiring information independently and attaining mastery of subject matter. Directed study recognizes and provides for individual differences, but it is not a strictly individual method. It aims to retain the advantages derived from mass instruction and at the same time to overcome the disadvantages arising from it. Since in mass instruction each individual cannot receive all the attention needed to help him understand and assimilate the principles of the course, directed study aims to provide for this attention by bringing the pupil into close personal relationship with the teacher.

The teacher's major task is to assist and direct the pupils, in particular to give to every individual direct instruction in the study habits which in his own case and in a special subject yield the best results. It is expected that by acquiring these habits the pupil in time will become intellectually independent.

The method has been advocated enthusiastically by teachers and administrators. It has been discussed widely in the educational literature. It was hoped by many enthusiastic teachers that the directed study procedure would solve all the problems and difficulties arising from group teaching. However, these results have not always been attained. In the opinion of some teachers, directed study is less satisfactory than the customary group teaching. They have found it more difficult to administer than the traditional procedure with which they were thoroughly familiar.

However, the question of efficiency cannot be answered by opinions alone. The fact cannot be overlooked that investigators uniformly report results in favor of directed study groups when compared with unsupervised groups.

The following represents a variety of claims made by a number of those whose experiments with directed study have been published: "It brought about general improvement in the whole school." "We used a 90-minute study period with excellent results." "The pupils preferred the new method to the traditional type." "It resulted in economy of time, increased efficiency, and a reduction in failures." "We observed an increase in interest in

studies. It reduced failures and decreased school costs." "We were able to excuse the bright pupils for part of the time."

The results of the following experimental studies are even more significant than the foregoing statements. In the University of Chicago High School[1] it was found that the poorer pupils profited considerably by supervised study, although the brighter pupils were somewhat retarded. Minnick[2] demonstrated that a supervised class surpassed an unsupervised class. Perhaps the most significant results are reported by Breed.[3] His experiment was conducted on a large scale, being carried on in a number of schools by teachers working under his direction. Furthermore, he defined the type of directed study he wished carried on, and worked out a definite technique to be used by those who co-operated in the experiment. He was thereby able to control certain conditions which made it possible to derive definite conclusions, at least regarding the type of directed study carried on in his investigations. The final outcome of his study may be summarized in the following main points:

1. In general, in all subjects, the poorer pupils made better progress than before, but the progress of the bright pupils was retarded. This might be expected. Directed study might easily neglect the superior student, just as the traditional recitation is likely to neglect the slow or less capable one.

2. In some subjects average results for whole classes showed an increase, while there was a decrease in others.

3. The efficiency of the divided- or double-period plans is in doubt.

4. The technique of directed study needs to be improved for the poorer pupils, and a new technique is to be developed for the brighter pupils.

Breed's results agree with former reports. The first and fourth points raise the question of providing adequately for individual differences; the second shows that the technique of directed study

[1] E. R. Breslich, "Teaching High School Pupils How To Study," *School Review*, XX (October, 1912), 505–15.

[2] J. H. Minnick, "An Experiment in Supervised Study," *ibid.*, XXI (December, 1913), 670–75.

[3] F. S. Breed, "Measured Results of Supervised Study," *ibid.*, XXVII (March, April, 1919), 186–204, 262–84.

must vary for various subjects, e.g., that directed study in mathematics may differ widely from directed study in English, or science; the third leaves open the question of proper division of time of the class period, which, after all, is not one of the principal factors. Breed's results are valuable because they indicate lines along which improvements may be sought.

How to provide for directed study in mathematics.—Many teachers are anxious to have directed study but are at a loss as to how to introduce it, or are afraid to introduce it because they lack an effective technique for conducting it.

In the following pages two ways of providing for directed study will be discussed. The first is to have supplementary study classes after school hours; the second is to use part or all of the class period for study. The first plan aims to help especially those who are unable to understand or finish the work during the class period, for those who have not applied themselves, and for those who have certain particular difficulties and need a little help or a few suggestions to enable them to proceed. Attendance may be required of pupils who have been absent, who have failed to do the class work, and who lack ability. Attendance may be voluntary for those who need help only on specific tasks.

The second method of having directed study is to make it a regular part of the class period, in which case the question of the amount of time to be used for it is an important one to decide. Some teachers favor a definite division of the class period, allotting a number of minutes to each of the various classroom activities. For example, in one school the following division of a 62-minute period was used: (1) recapitulation, 5 minutes; (2) statement of the business of the day, 2 minutes; (3) discussion, 25 minutes; (4) assignment, 5 minutes; and (5) study period, 25 minutes.[1]

An arbitrary division of class time for study or any other purpose is an effective administrative device of securing co-operation on the part of every teacher, especially those who are not interested. Pedagogically, it is unwise to set arbitrary time limits for teaching, study, reciting, or any other class-period activity. In general, it is better to let them follow one another in natural order. It might

[1] Eula Young and M. R. Simpson, "A Technique for the Lengthened Period," *ibid.*, XXX (March, 1922), 199.

be to the best interests of the class to teach and reteach a lesson before attempting study. Sometimes it might be advisable to use the whole period for presentation and teaching purposes. At other times one or more periods may be needed for continuous uninterrupted study. It is a mistake to curtail the teaching time at a moment when the class is just beginning to understand, or to interrupt assimilative study too soon. The disadvantages of an arbitrary time limit more than offset administrative advantage.

The technique of directed study.—The technique of directed study in mathematics is really very simple. Let us assume that the teaching has been done and that a brief oral test has determined that the pupils have caught the idea and have grasped the principle to be learned. The class is then ready for study and begins work on an assignment. The teacher, standing at the desk, waits perhaps a minute or more until every pupil has started. When all are at work, he passes quietly down the aisles to make a first survey of the situation, noticing particularly what the slow pupils are doing but not stopping to give individual help. If he finds that many pupils, besides the few who always have difficulty, are unable to start the work; or if he discovers that some unexpected difficulty or misunderstanding interferes with progress, all pupils should be asked to give attention to the instructor. Further teaching may have to be done, or it may be sufficient to offer a few suggestions and some additional directions. Even the simplest directions are sometimes misunderstood, in which case the pupils lose time vainly trying to start the work before the teacher realizes that there is a general difficulty.

Failure to make the first survey might be the cause of considerable waste of time. To illustrate, a teacher gave an assignment in which the first step was "draw a right triangle having one of the acute angles equal to 30 degrees." The pupils worked busily. Some time later, when the teacher thought that the pupils should have completed the solution of the first problem, he asked those who had finished to raise hands. There was no response. Since everyone seemed to be working industriously, the teacher decided to give them more time, but soon it became apparent that the pupils were not getting results. They looked puzzled. The teacher then walked down the first aisle, looking at the work of some of the pupils. He

discovered the cause of trouble immediately. The class had been studying angles for some time, and the pupils were confusing right "triangle" with right "angle." Instead of a right triangle they had drawn a right angle, which they divided into two acute angles, one of them being 30 degrees. They could not go on with the remainder of the problem. An early survey would have detected the difficulty at once.

There are other reasons for making this first survey. For example, pupils settle down to work quickly and apply themselves readily if they know that their work is soon to be examined by the teacher. This is the teacher's opportunity to help them develop the habit of starting promptly.

The first survey is at once followed by a second. The teacher now passes from desk to desk and gives assistance to those pupils who generally have difficulties. He pays little attention to those who need no help.

Throughout the remainder of the study period the teacher makes brief comments to the pupils. Some are encouraged, some are praised, others are criticized for bad habits of study, careless written work, or uneconomical use of time. Each pupil has the chance to attain complete understanding. Whenever it is found that a mistake is made by a number of pupils, needless repetitions are avoided by stopping all work and explaining the difficulty to the entire class.

As simple as the technique of directed study may seem, it raises several serious problems for the teacher. Success of the method is more or less endangered unless careful attention is paid to such points as the following:

1. Study should not begin until evidence has shown that the assignment and principles are sufficiently understood to enable the pupils to work independently. If they have so many questions that the teacher cannot answer them conveniently and if too many pupils do not know how to proceed, this probably indicates that the class is not yet ready for study and that more teaching should be done.

2. Each pupil must be provided with the complete equipment needed in the work to be done. Otherwise, when study begins, there will be interruptions and delays. Pupils should never be permitted

to borrow eraser, compasses, ruler, or protractor, during the study period. Whenever materials are borrowed, two pupils are disturbed. In some cases this may become the cause of disorder and interfere with the work of the teacher, whose undivided time and energy should go toward helping individuals.

3. The teacher must not become too much engrossed in the efforts of one pupil, or of a few pupils, while others waste time and the class suffers from inattention. The interests of the class cannot be subordinated to those of one or of several pupils.

4. Pupils must not get too much help or they will become dependent on the teacher, which is exactly the opposite of what directed study aims to accomplish. It is poor policy to make the same suggestion to a pupil day after day unless he is thereby stimulated to help himself. Directed study must arouse self-activity on the part of every member of the class. The study-room situation should always resemble the home-study situation, where the pupil is alone with his books and with no one to appeal to for help. To be sure, in the classroom the teacher is present to direct and assist; but he is the one to decide when help should be given, not the pupil. The pupil must not sit idle waiting for the teacher. He must be taught the habit of keeping himself profitably employed at all times.

5. Special devices must be worked out for giving help. If the pupil's difficulty is due to poor reading, he may be asked to read again more carefully, to reconstruct the sentence, or to look up the meaning of a difficult word. The easy way of giving straightforward replies to all questions and of telling him everything should be avoided. Another question may be proposed or a suggestion made. The pupil must be given time to think. Only when it becomes apparent that he has tried and that with the best effort he cannot do the work, does he deserve more help than a question or a suggestion. Sometimes the teacher may have to tell the answer outright so he may go on. We cannot expect the impossible. It is just as fatal not to tell a struggling pupil enough as to tell him too much.

It is hardly necessary to add that all help should be given in an undertone in order that the other pupils are not disturbed.

6. Unusual pupils will be found even in classes formed on the basis of ability. Devices must be developed to take care of individual differences and to pay attention to individual needs. The bright

pupils may be allowed to proceed more rapidly or may be given a deeper course than the others. They may be given supplementary work when they have finished the assignment, or they may be released from classwork until the class as a whole has caught up with them.

The teacher must be on guard not to conclude too quickly that a pupil is bright. He might be only a rapid worker, a lesson-learner rather than a thinker. He might memorize without really learning. When a difficulty appears, he does not try to overcome it but hurries on to the next problem. His principal aim is to show off before the class and to make a high record in the number of problems finished. Even the brightest pupil may lack important study habits and may need to improve in others.

At the same time many pupils regarded as slow are doing some careful thinking. With a little individual help and sympathy they can soon be put on the road to success. Individual differences in results are magnified by the difference in former experiences and training. It is possible to minimize these differences by careful supervision and by paying more attention to individual needs. All pupils must be helped to grow in ability to study.

7. Directed study, if properly conducted, is a mental and physical strain on the teacher. The progress of every pupil has to be watched and kept in mind; everyone's needs have to be attended to. Hence, when it is found that everyone is working satisfactorily, the teacher may be seated at his desk and the few pupils who need help may come to him, one at a time, for suggestions. At no time during the study period should a teacher busy himself with work other than that of directing pupils. The teacher who uses it to dispose of minor routine tasks will accomplish but little.

8. Directed study does not necessarily involve the abandoning of home study. It should simplify it and reduce the amount, and it should improve the character of work assigned to be done at home. Home work, to be effective, must not be assigned until the teacher has evidence that the class is prepared to do it. Any other method is unfair, wasteful, and cannot lead to intellectual independence.

Directed study gives excellent training to the teacher.—Lack of knowledge as to the difficulties of a course has always been the cause of much unsuccessful teaching. During directed study they

are bound to come to the surface. Through actual observations the teacher learns how pupils study. With concrete evidence constantly before him, he soon knows how to anticipate difficulties and improves his methods of instruction to enable pupils to learn most economically. He learns to give directions which all pupils can understand.

The following illustration taken from actual class observation shows how directed study reveals conditions which are not usually discovered when the pupils do all their studying at home. The observation was made in a class conducted by an experienced teacher who had just taught a junior high school class how to solve a type of equation of which $\frac{x}{3}+\frac{x}{5}=8$ is an example. After developing the method of solving the equation, the teacher emphasized the following steps which the pupils were to take in the assignment to follow:

1. Multiply each term by the least common multiple of the denominators:

$$\frac{15x}{3}+\frac{15x}{5}=15\times 8$$

2. Reduce the fractions:

$$5x+3x=120$$

3. Collect similar terms:

$$8x=120$$

4. Divide both sides by 8:

$$x=15$$

5. Check the result:

$$\frac{15}{3}+\frac{15}{5}=8$$

$$5+\ 3=8$$

After further questions asked by pupils had been discussed by the teacher, the assignment was made, in which the first problem was to solve the equation: $\frac{x}{4}-\frac{x}{7}=6$. Disregarding any occasional mis-

takes, the observer noted the following, because each was made by a number of pupils.

1. Several pupils wrote: $\frac{15x}{4}-\frac{15x}{7}=15\times 6.$

Apparently, the pupils making this mistake did not understand how the least common denominator "15" had been derived in the illustrative example, and therefore could not derive it in the assigned problem. They looked at the illustrative problem which was still on the board, noted that the first step was to multiply each term by 15, and did the same in the assigned problem. Evidently they were not ready for study. They needed further illustrations to make the first step clearly understood. They tried to imitate the teacher. They merely looked to see what was done in the example and did identically the same thing with the assigned problem.

2. $\frac{28x}{4}-\frac{28x}{7}=28\times 3.$

Again the tendency to imitate the illustrative problem was apparent. These pupils were not able to tell where the "8" in the right member of the example came from. They decided that it was the result of adding the denominators. Hence, in the assigned problem they wrote the difference: $7-4$, or 3, in the corresponding place.

3. In the solutions of several pupils who had taken the first step correctly, $7x-4x=168$ was followed by $11x=168$.

In this case the pupils again imitated the illustrative example, in which the $8x$ was found by addition, but disregarded the minus sign in the second problem and performed the operation which was used in the first. Confusion of operations is one of the most common errors in algebra and arithmetic.

4. As a check, several pupils wrote the following: $\frac{28}{4}-\frac{28}{7}=6.$

This error may be explained as follows. Instead of using the "solution" 56 in the check, the pupils used the "multiplier" 28. Since in the illustrative example the "solution" and the "multiplier" both happened to be 15, it made no difference whether they used the solution or the multiplier in the check. It made a great deal of difference in the assigned problem, however.

Without directed study these mistakes could not have been ob-

served. After teaching how to solve the equation, applying the method to a specific problem, and assigning for home study further problems to be solved in a similar way, the teacher's work would have been finished. The pupils, after wasting some time, would have received more or less help somewhere, and would have brought in the correct work the next day. The papers would have led the teacher to believe that the pupils had attained mastery of the process without difficulty. Directed study, however, brought to the teacher's attention two serious pedagogical errors. First, she had failed to obtain evidence to show that the pupils were ready for study. If she had sent the class to the board to solve the equation $\frac{x}{4}-\frac{x}{7}=5$, several difficulties would have appeared immediately and the need for further teaching and additional illustrative exercises would have been recognized.

Second, it was a mistake to leave the illustrative example on the board. This induced the pupils who did not understand the solution to imitate what seemed to have been done in the first exercise. Pupils should not be allowed to copy solutions while a process is being taught, and illustrative problems should be erased from the blackboard before study begins. This practice will develop a greater feeling of responsibility and a higher degree of concentration among pupils when they are listening to explanations.

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TEXTBOOKS ON DIRECTED STUDY

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The blackboard as a teaching device.—In chapter i reference was made to the use of the blackboard as a means of determining quickly and objectively whether or not the class is ready for an assignment. The information might have been secured by letting the pupils work at their seats, but the teacher would then have to examine each pupil's work separately. The board method has an advantage over the seat work because it shows at a glance what progress is made by the class as a whole. It is economical as long as all pupils are profitably employed.

The blackboard is very convenient for conducting certain phases of directed study. After the assignment is made, the pupils write all of their work on the board. With the written work before him the teacher can easily identify those pupils who have difficulty and need further directions. Passing from one to the other, he gives whatever assistance is needed.

The blackboard is also well adapted to rapid-fire drill. As the teacher reads the problems, one at a time, the pupils work them as rapidly as they can.

During the supervised study period individual pupils may be sent to the board to write out solutions of problems which the class as a whole has been unable to work. Sometimes several pupils may write their solutions on the board to illustrate different ways of solving the same problem. While these pupils are writing on the board, class study should not be interrupted, and it is not always

necessary to have the solutions explained. The other pupils may examine them when they find it convenient to do so.

The use of the blackboard has its limitations. Much of the board work observed in our schools is uneconomical and therefore to be avoided. Thus, one often sees some of the pupils of a class working at the board while the others are waiting idly for them to finish because the teacher failed to give them something definite to do. It is an infallible rule of good class work that every pupil must be profitably employed at all times. Either he must keep himself thus employed or a special assignment should be made.

Another wasteful blackboard procedure which is very common is the sending of pupils to the board to work out problems which they explain later. The teacher, not the pupil, is the one to explain the problems to the class. For, a pupil's explanation of a difficult problem is rarely satisfactory for those who failed to do it, and the pupils who did understand it are not interested in the explanation. It will be argued by some that the pupil, in explaining a problem, receives valuable training in oral expression. The answer is that such training should be given by procedures which do not waste the time of the class.

The textbook as a teaching device.—The textbook in mathematics is a most valuable aid to teaching. It supplies the necessary subject matter, contains suitable exercises and practice materials, and suggests local materials to be gathered by the teacher or the pupils. Moreover, it shows how to organize the various units and topics. It is a time-saver to the teacher. On the other hand, it enables the pupil to study further what he failed to understand in class, and to make up losses due to absences or other causes.

Without a textbook much time is needlessly spent in dictating and copying. Furthermore, the pupil is deprived of exceedingly important and valuable training in deriving thought from the printed page. It has been found that classes taught by the no-textbook method are handicapped in the succeeding courses because they do not know how to use a textbook as well as their classmates.

Objections to the use of the textbook will cease when the teacher has learned to adapt textbook materials to the abilities, interests, and needs of the pupils. No textbook alone, however well it may be written, can replace the teacher. It merely supplements instruc-

tion. Hence, the practice of regularly assigning arbitrary lessons of textbook materials to be followed the next day by formal recitations is to be condemned. It trains the pupil to accept facts blindly and to memorize large amounts of materials that have no meaning to him. He learns parts but fails to see their relations to the whole. He cannot judge the relative values of textbook materials.

The foregoing deficiencies are not to be regarded as arguments against the use of textbooks in mathematics but in favor of the correct use. They suggest more frequent use, and call attention to the need of systematic training to enable pupils to read the textbook with understanding and to do additional reading in other books. The problem of developing good habits of reading is to be more fully discussed in chapter iv.

Practice in mathematics.—Drill differs from practice in the sense that it aims to attain by mechanical repetition a degree of familiarity which should enable the pupil to respond automatically and to retain facts permanently. Practice is a part of the process of assimilation, while drill is to be given only after evidence has shown that understanding has been attained. We cannot teach by drill.

There are relatively few situations in secondary-school mathematics where drill is desirable or necessary. After the sixth grade drill becomes less important, and more emphasis is to be placed on thinking in terms of mathematical laws and principles and on understanding.

Although practice may be administered to the group as a whole, it must reach the individual. He must understand the purpose clearly. His interest must be aroused and retained. One of the best devices of keeping him interested is competition. Pupils like to surpass standards or the records of their classmates or their own records. However, pupils are apt to tire, and the teacher should be on guard not to continue it after the point of fatigue is reached. Special attention should be given to difficult cases, as, for example, the zero difficulties in subtraction and multiplication.

Oral class practice.—This consists of many exercises, each pupil being held responsible for the correct answer to every question. The exercises should be simple in order that the pupils may make actual use of the principle involved in them and not lose sight of it in a mass of details, which happens often when complicated exer-

cises are presented too early. We must not slight the easy exercises in giving practice in the application of mathematical principles.

In algebra this type of practice calls for rapid work in such processes as the fundamental operations and factoring; in solving simple equations and problems; in stating rules or steps in important processes; and in translating verbal statements into symbols. In conducting oral practice the teacher may call upon individual pupils in the order in which they are seated; but she should realize that while the device makes certain that every pupil receives practice, it might lead some to relax until their turn comes to respond. Usually, therefore, the order should be varied, in which case the teacher must watch carefully that the pupils are called upon an equal number of times. Otherwise, the aggressive individuals may be the only ones to derive practice. Most modern textbooks contain a sufficient amount of practice material that may be used to supply the needs for this type of work.

In geometry, oral practice is less important than in algebra or arithmetic. It may consist of stating rapidly the theorems learned; translating verbal geometric statements into symbolic forms; naming the steps in certain methods of proof, e.g., in the indirect method; and giving summaries and classifications of principles, e.g., naming all the conditions under which lines are equal or parallel, and triangles are similar or congruent.

A very effective practice device in mathematics is the mathematical contest. The class is divided into two teams, either of which tries to surpass the other in solving correctly a greater number of problems (see chap. iii).

Written class practice.—As in the oral work, this type of practice aims to develop speed and accuracy through the increased effort which comes from competition, but it has the additional advantage of supplying written material which may be used for diagnostic purposes. It might even be used as the basis for remedial work. Written practice may be conducted in several ways. The whole class may go to the board, or some pupils may remain seated while the others go to the board. A problem is then assigned to all. When a number of pupils have finished, all work is stopped; and the first, or the best, solution is explained by a pupil or by the teacher. A serious objection to this type of practice is that some pupils find the

time too short to finish, and therefore have incomplete experiences. For this reason it should not be used too often.

Another way of conducting written practice is to assign to each row a different problem and to have that pupil in a row who is the first to finish write the solution on the board and later explain it to the class.

Finally, written practice may be given in the form of printed exercises. Interest and attention may be stimulated by setting time limits for the purpose of determining how many exercises each pupil can work correctly in the given time. When the exercise is repeated, each pupil is urged to excel his former record.

The resourceful teacher will find other ways of conducting practice. It is well to vary the types of practice when interest diminishes. Competition and the play spirit will preserve interest and attention and will urge each pupil to do his best.

A teacher who is unable to secure published practice material should have little difficulty in preparing his own. The following examples illustrate suitable materials for written practice. The time limits are to show the pupil what he might reasonably be expected to do. Overemphasis on speed might be the cause of inaccuracy.

Such practice material might also be used as test material, the pupil testing himself until he feels satisfied with his accomplishment.

Example 1:[1] Combine similar terms, doing as many of the problems as you can in 2 minutes. To save time, place a sheet of paper to the right of the problems and write only the answers:

1. $3x+5x+8x-2x$
2. $2a+3a-a+a$
3. $6b-2b+4b-b$
4. $5a+a+10+3a-6$
5. $9a+13b-4a-b$
6. $3x+4y+7z-x-y$
7. $m+2n-3t+8m-t$
8. $15a+3+12a-1-5a$
9. $5x+5y+z+3x-y$
10. $4a+2b+c-3a-b+5c$
11. $x+3y+8+y-x-3$
12. $10m+4r-t-4r+t-10m$

[1] E. R. Breslich, *Senior Mathematics*, Book I, pp. 22, 23.

13. $r+7w+4s-3w+4r+6s$
14. $5x+8y+9+3x-7y-6$
15. $2x+5y+10+4x-3y-7$
16. $4a+b-4c+a+6b+6c$
17. $x+7y+z+3x-y-z$
18. $a+9b+5+a-7b-2$
19. $3x-8y+4z+3x+8y-2z$
20. $5a-7b+18+6a+8b-16$
21. $10x+3y+(5x-y+9)$
22. $4a+b+8c+(a+2b-3c)$
23. $m+6n+5t+(4m-2n-t)$
24. $3x+5y+12+(x-4y-6)$

Try the test again and see if you can increase your score.

Example 2:[1] In the following problems do as much as you can in the time indicated. Use pencil only when necessary. Write your answers on a sheet of paper placed to the right of the equal signs. Take the tests a second time to see if you can increase your score.

I. Time: 2 minutes.

1. $a+7=12$	$a=$
2. $b-14=2$	$b=$
3. $x-9=6$	$x=$
4. $3x+5=17$	$x=$
5. $2a-6=14$	$a=$
6. $3t+7=28$	$t=$
7. $9a=3a+30$	$a=$
8. $7x=2x+20$	$x=$
9. $6x-4=2x+9$	$x=$
10. $4m+3=m+11$	$m=$
11. $4b+5=8-b$	$b=$
12. $8x-22=3-5x$	$x=$

II. Write the equation for each of the following, but do not solve.

Time: $2\frac{1}{2}$ minutes.

1. The sum of two numbers is 64. The greater is 3 times as large as the smaller. Equation:_____

2. A farmer wishes to raise 7 times as much wheat as corn. If he has 280 acres, how many of each should he plant? Equation:_____

3. A farmer divides a 550-acre farm among three sons, giving to the youngest twice as many acres as the oldest, and to the second as many as the oldest. Equation:_____

[1] *Ibid.*, p. 62.

4. The length of a rectangle is 6 feet less than the width. Find the dimensions if the perimeter is 98 feet. Equation:_____

5. What sum of money left at 5 per cent simple interest for 2 years will amount to \$348? Equation:_____

Example 3:[1]

1. Add the lower number to the upper. Time: $1\frac{1}{2}$ minutes.

$10a$	$-6a$	$-26x$	$+14b$	$-1.2x$	$+6x$
$-3a$	$+4a$	$-13x$	$3b$	$+4.1x$	$-11x$
0	$-4a$	x	$.43m$	$\frac{1}{4}x$	$-\frac{1}{5}n$
$+6a$	$+0$	$.75x$	m	$-\frac{2}{5}x$	$\frac{2}{3}n$

2. Subtract the lower number from the upper. Time: $1\frac{1}{2}$ minutes.

$-11a$	$10x$	$-8b$	$-22a$	c	$-19x$
$3a$	$-11x$	$5b$	a	$-18c$	$-x$
$-m$	$-15c$	$2a$	$\frac{2}{5}r$	$\frac{3}{5}x$	$-\frac{5}{6}m$
m	$-c$	$-a$	$-\frac{1}{2}r$	$\frac{2}{3}x$	$-\frac{1}{3}m$

3. Multiply and divide as indicated. Time: 2 minutes.

$-4 \cdot 3b$	$20a \div -5$
$-5 \cdot 6a$	$-18x \div +3x$
$8 \cdot -5r$	$+2a \div +a$
$-2 \cdot -4b$	$-5a \div -2$
$-12 \cdot +x$	$-8x \div +x$
$+3b \cdot -2$	$-14b \div +2b$
$2a \cdot -4$	$+16 \div +4a$
$-a \cdot -6$	$+12 \div -3x$
$-5a \cdot 0$	$-3 \div +5a$
$0 \cdot -4c$	$-4x \div -2$
$-1 \cdot +6x$	$+6y \div -3y$
$3a \cdot -1$	$20x \div -x$

The use of projects.—As the name indicates, the "project method" aims to teach through an enterprise broad enough to interest a large number of pupils. It is to be a constructive piece of work, more than just recreation. It is presented in a natural setting and employs natural methods of learning. Some of the major purposes

[1] *Ibid.*, p. 86.

are: to adjust the boy and girl to the environment in which they live, to connect the subject with the lives of the children, to make contacts with adult life, and to offset the arbitrary divisions, topics, and lessons existing within and among subjects. The following are good examples of projects for seventh-grade pupils: building a home, making a garden, running a school bank or store, planning the family budget, raising poultry, simple surveying, and designing. A little reflection shows that such projects offer large possibilities for teaching and learning mathematics and for correlating mathematics with other school subjects.

Objections to the method arise when teachers try to make it supplant rather than supplement other methods. Furthermore, not many teachers have the range of knowledge required to help pupils carry their projects to a successful completion. It takes much ability and skill to use the method successfully. Frequently the parents object to the method because it makes too great a demand on the pupil's time outside of school. Moreover, when a project extends over too long a period, the pupils, although their interest was at first keen, eventually tire, and the project loses one of its greatest values, namely, that of motivating study and of keeping the learner interested.

On the other hand, the method assists the teacher in discovering pupils whose interest in certain phases of the project is genuine and who are anxious to go into it more deeply. Here a real opportunity is presented to stimulate independent study, which might lead the pupil to take up an individual project.

It is not to be expected that all pupils take up a special project or that all of those who undertake one will finish it. Nor is the project limited to the mentally superior pupils. Anyone who cares may engage in this undertaking. It is quite common that the feeling of success which comes from the completion of an independent piece of work strengthens a pupil whose class work is only of mediocre character.

Sometimes the teacher suggests a topic. Suppose the class has shown interest in the metric system. It is then quite proper for the teacher to say that anyone interested in further study of that phase of the work will find considerable reading material in several of the books available in the mathematical library and that such a study

would make a satisfactory project. Although the project is suggested by the teacher, the pupil who takes up a project must make it his own. Otherwise, one of the principal values of the method will be lost.

An interesting graph taken from a newspaper may arouse discussion which in turn may lead to a project on graphs. An excellent design exhibited on the bulletin board might inspire some student to take up a project of that type.

However, it is best that a pupil find his own project and not to have one suggested. Pupils who finish a unit of work ahead of the class may be excused to go to the mathematical library for reading and study. There in the mathematical literature they may find a project of sufficient interest to arouse a desire for further reading or study.

To facilitate the transfer of students from the classroom to the library, the department of mathematics of the University of Chicago High School has designed the blank shown below for the use of those who are excused from class work to study in the library.

University High School

Department of Mathematics

Instructor_____

SUPPLEMENTARY READING REPORT

Name of pupil:_____ Math.:_____

Date:_____ Period:_____ Time spent:_____

Author:_____ Title:_____
(Book or magazine)

Topic:_____

Chapter:_____ Pages:_____

Report:

The blank is of No. 6 notebook size, and the lower part allows enough space for a written report on the reading done by the pupil during the time spent in the library.

When a supplementary project is finished, the student should put it in writing in clear and attractive form. Such a paper should not be essentially different from the customary term paper of the college student, and, being a voluntary task, it frequently will be superior. The following outline is suggested: Title of paper, table of contents, brief statement of the problem, organization and treatment of the study, summary and conclusions, bibliography. Some pupils will report their projects to the class and thereby receive practice in oral expression. In general, however, a finished project will be delivered to the teacher in written form.

It should be kept in mind that an individual project is not an assigned requirement of the course and not something to be done for credit. It is taken up by the student because he enjoys it. He finds it valuable to himself because it leads to interesting information, offers excellent training in developing good habits of study, acquaints him with the library and its uses, and establishes close relationship with the teacher, since the latter will be frequently consulted for advice and suggestions.

Literature relating to the project method.—The reading of some of the following references will acquaint the reader more thoroughly with the use of the project method.

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Jablonower, Joseph. "The Project Method and the Socialized Recitation," *Mathematics Teacher*, XXI (December, 1928), 431–41.

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Summary of chapter ii.—The suggestions, devices, and procedures discussed in chapter ii may be briefly classified as follows:

1. A good teaching procedure must reach pupils individually even though it might engage the group as a whole.

2. There is no single procedure which surpasses all others in attaining superior results under all possible conditions. The teacher must familiarize himself with several of the best and adapt them to the various teaching situations.

3. The following procedures have been discussed:

The lecture method
The drill method
The genetic method
The heuristic method
The question-answer method
The directed study method

4. The chapter has also called attention to the advantages and disadvantages of the following devices:

The blackboard as a teaching device
The use of the textbook
The use of practice materials
The use of special projects

5. Directed study offers excellent opportunities for selecting the methods and devices best adapted to various teaching situations. Definite suggestions have been presented to assist the teacher in conducting directed study.

CHAPTER III

AROUSING AND MAINTAINING INTEREST IN MATHEMATICS

Motivating the study of mathematics.—It is a well-known fact that superior results are obtained when the pupil is genuinely interested in the work to be done. On the other hand, failure to show the purpose of a study often results in lack of interest and even in defiant opposition of the pupil. An example of this was observed during a visit of a class in algebra. It was the first meeting of the course, and there were about forty pupils in the class. The teacher, a man of many years of experience, conducted the work admirably. At the end of the class period a few minutes were left over with nothing definite to do, and the pupils proceeded to ask some questions. One girl raised her hand and asked, "Why should girls study algebra?" After a moment of reflection the teacher said, "You must study algebra because it trains your mind." Apparently not satisfied, the girl raised her hand again and asked, "But there are other subjects which we may study to train our minds, why should it be algebra?" This time the teacher answered, "You should study algebra because it is a useful subject and some day you may need it." Again the girl spoke: "Where does a girl ever use algebra?" The teacher now showed signs of irritation, for the class had begun to enjoy the situation and seemed to take the part of the girl against him. His reply was sharp and final: "You must study algebra because the Board of Education of the city demands it." No further questions were asked. The argument was closed, but the class felt that no valid reason for the study of the subject had been given and that probably none existed. It was evident to the observer that the teacher knew his subject, and that he knew how to teach it, but that he had never formulated a statement of the purpose of the study which the pupil would accept without further question. Hence, some of his pupils began algebra with an attitude that was passive. Others were clearly hostile. The whole difficulty could

have been avoided if the teacher had told his pupils at the beginning why they should study algebra. He might have started the work with a preview presenting a few definite motives which the pupils would recognize as worth while. He might have invited their interest by telling them something about the activities they were going to perform. The question "Why should girls study algebra?" would not have been asked. With a grasp of purpose and meaning would have come good will, interest, attention, intelligent co-operation of the pupil, and a desire to learn. It is one of the primary duties of the teacher of mathematics to make the subject interesting and to motivate the pupil activities.

One frequently hears the statement that high-school girls have more difficulty with mathematics than boys. The implication is that boys have a special ability which is lacking among the girls, but the real explanation is probably the fact that in the past mathematics textbooks have addressed themselves mainly to the boys. The subject has been related to phases of life which appeal to boys, while those in which girls have a strong interest have been neglected. Applications to mechanics, science, and engineering are mainly of interest to boys, but applications to designing, dressmaking, home-furnishing, and household, are better adapted to the interests of girls and should therefore be stressed in modern teaching. Any method or procedure is greatly influenced by the extent to which it arouses and holds the interest of the pupil. Indeed, interest changes into pleasure work which otherwise might be drudgery.

Literature relating to motivation.—The following references contain many valuable suggestions as to ways of motivating the study of mathematics.

Anonymous. "Flatland," *School Science and Mathematics*, XIV (October, 1914), 583–87.

Baker, H. B. "Animated Mathematics," *Mathematics Teacher*, XVII (December, 1924), 482–85.

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Carmichael, R. D. "Mathematics and Life—The Vitalizing of Secondary Mathematics," *School Science and Mathematics*, XV (1915), 105–15.

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Hedrick, E. R. "The Significance of Mathematics," *Science*, XLVI (October 26, 1917), 395–99.

Herrick, C. A. "What High School Studies Are of Most Worth?" *School and Society*, IV (August 26, 1916), 305–9.

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Rabourn, Sara. "A Few Classroom Devices To Stimulate Interest in Mathematics," *ibid.*, XX (October, 1927), 328–33.

———. "Boost Mathematics," *School Science and Mathematics*, XVI (October, 1916), 595–602.

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Romberg, Hildegarde. "Presentation of the Addition of Positive and Negative Numbers by Motivations," *School Science and Mathematics*, XXVII (February, 1927), 155–58.

Sampson, C. H. "Geometrical Movies," *Journal of Education*, XCIV (October 6, 1921), 328.

———. "Filmed Geometry," *Journal of Education Method*, I (November, 1921), 116–17.

Schlauch, William S. "An Experiment in Motivation," *Mathematics Teacher*, XII (September, 1929), 1–9.

Shaw, J. B. "Teaching Mathematics to Girls," *ibid.*, XVIII (December, 1925), 455–65.

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Statham, J. F. "Is Geometry Possible?" *ibid.*, XXI (October, 1928), 353–56.

Stroup, P. "Direct Cultural Motivation for Demonstrative Geometry," *ibid.*, XX (March, 1927), 173–77.

Waples, Douglass. "Teaching Teachers How To Motivate," *Educational Administration and Supervision*, VII (November, 1921), 439–46.

Using the social values of mathematics as a motive for study.—The social uses of mathematics serve as excellent motives for the study of the subject. The pupil should be made to feel that the subject has real values in the affairs of everyday life. For example, in introducing the study of graphs, the teacher may bring to class several striking graphs clipped from recent issues of newspapers or magazines. A discussion of one graph taken from the morning paper makes a far greater appeal to a class than a number of graphs taken from the textbook, because it is up to date and therefore worthwhile material. When pupils see that the graphs which they are about to study are actually used by business concerns, manufacturers, and statisticians, no further evidence will be needed to convince them that everybody should know how to read, interpret, and even make graphs. For the same reason local material brought in by the pupils themselves should replace much of the problem material of the textbook. To the resourceful teacher the textbook merely suggests what should be done. He will seize every opportunity to emphasize the social values of the various units of the course, thereby leading the pupil to form the growing conviction that mathematics is an important aspect of the social life of the community.

Showing the practical phases of mathematics.—The practical applications of mathematics are not always directly useful to the pupil, but they help to make the subject seem real. No opportunity should be lost to call attention to genuine practical applications wherever they occur in the study of mathematics. The following examples illustrate how this may be done:

a) Using practical applications as a means of introducing a new unit.— The start of the unit often determines the extent to which the pupil will take an interest. The teacher should do his best to introduce the unit in a way which will make it seem attractive to the pupil. Thus, in the beginning of the study of parallelograms he might illustrate the practical uses of some of the principles to be learned in the unit by discussing several mechanical devices whose constructions are based on these principles. For example, most pupils have observed that a little shelf like the one illustrated below,[1] when moved about in space, always remains in horizontal position. The reason is easily explained by means of one of the theorems taught in the unit on parallelograms.

The construction of the parallel ruler, which is used by draftsmen to draw parallel lines, is also based upon a principle of parallelograms.

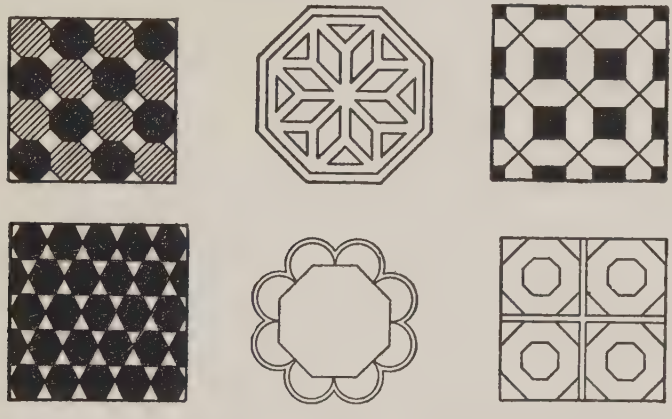
The familiar device by which the telephone is drawn up or pushed out of the way without changing the vertical position, is a practical application of the fact that when the opposite sides of a quadrilateral are equal they are also parallel.

Illustrations like the preceding motivate the study of units which otherwise might seem abstract, remote from life, and therefore uninteresting to the pupils. When interest is aroused, the study of the unit will be a combination of pleasure and work.

Motivation of the unit on regular polygons is illustrated as follows:[2]

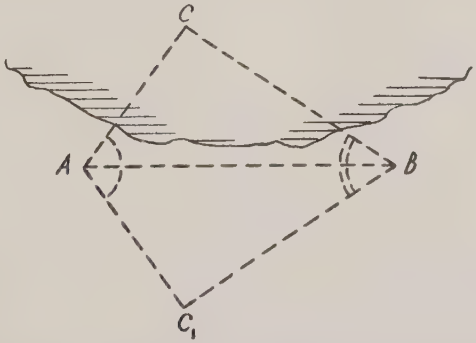
[1] E. R. Breslich, *Senior Mathematics*, Book II, p. 70. [2] *Ibid.*, p. 220.

"Regular polygons are involved in many forms of decorative design. You have observed designs like those illustrated on this page in tile floors, ornamental windows, linoleum patterns, paper doilies, ceiling panels, floor borders, and furniture designs. Try to find other designs using regular polygons, and bring them to school to show them to the class."



b) Motivating the study of mathematics by means of practical applications of the principles taught in a given unit.—The following shows how the surveyor makes use of a theorem on congruent triangles.[1]

"Let it be required to find the distance AC, C being a point in a lake and A being a point on the shore.



"A base line, AB, of convenient length is laid off, and the angles at A and B are measured with a surveyor's transit.

[1] *Ibid.*, p. 21.

"Then AC_1 is laid off, making $\angle C_1AB = \angle CAB$; and BC_1, making $\angle C_1BA = \angle CBA$.

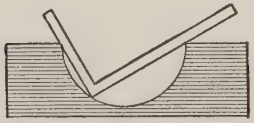
Finally AC_1 is measured.

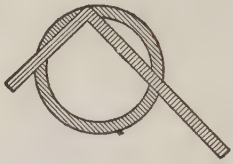
Show that $\triangle C_1AB \cong \triangle CAB$, since two angles and a side of one triangle are equal respectively to the corresponding parts of the other. It follows that the length of AC_1 is also the required length of AC. For corresponding sides of congruent triangles are equal."

There are few applications that appeal as strongly to both boys and girls as the simple problems in determining unknown distances. If no surveyor's instruments are available, homemade instruments, such as described in chapter v, will fully answer the purpose.

A few other practical uses of mathematical principles are shown on the following pages. The first is taught in the unit on inscribed angles.[1]

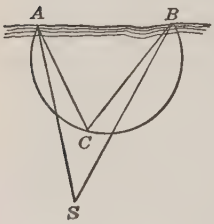
1. "Show how a carpenter's square may be used to test the accuracy of a semicircular groove."





2. "Show how a carpenter's square may be used to find where a ring must be cut so that the two parts are equal."

3. 'The circle in the adjacent figure includes a region of dangerous rocks to be avoided by ships passing near the coast AB. Outside the circle there is no danger. Show that the ship S is out of danger as long as the angle ASB, found by observations made from the ship, is less than the known angle ACB."



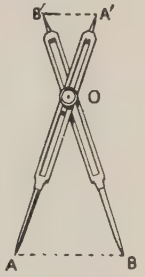
The following illustrations show uses of several principles taken from the unit on proportional line segments.[2]

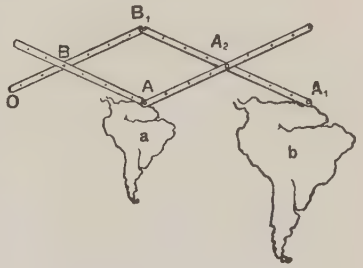
4. "The adjacent figure represents a pair of proportional compasses. It is used to make scale drawings of given figures. The instrument consists of two intersecting lines AA' and BB', that can be clamped together at the point of intersection, O, by means of a screw. By making $OB' = \frac{1}{2}OB$

[1] *Ibid.*, p. 199. [2] *Ibid.*, pp. 92, 93.

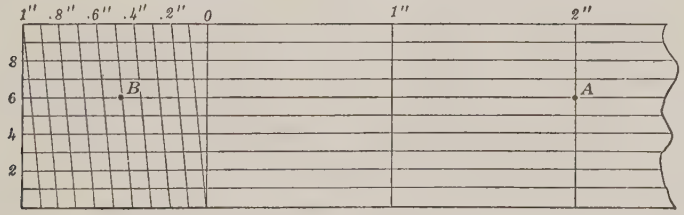
and $OA' = \frac{1}{2}OA$, and by opening the compasses so that AB is equal to a given line segment, you obtain $A'B'$ equal to $\frac{1}{2}AB$. This fact follows from one of the principles of proportional line segments."

5. "The pantograph is an instrument used to draw figures to definite scales, and to enlarge or reduce maps, drawings, designs, etc. It consists of four bars, fastened together in such a way as to make BB_1A_2A a parallelogram. According to certain principles of proportional line segments, if $\frac{OB}{OB_1}$ is made equal to $\frac{B_1A_2}{B_1A_1}$, points O, A, and A_1 must fall in a straight line and make $\frac{OA}{OA_1} = \frac{OB}{OB_1}$. Keeping point O fixed and making point A describe figure a makes the pencil point A_1 describe figure b. The two figures are similar in shape. In fact, figure b is really figure a *magnified* to the scale OB_1 to OB."





6. "The diagonal scale is another instrument whose construction is based upon principles of proportional line segments. By means of it lengths may be measured to hundredths of an inch."



Literature relating to the social and practical uses of mathematics. —The following references have been found helpful to teachers in familiarizing themselves with the social, vocational, and practical uses of mathematics.

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Harold, H. R. "A Study of the Mathematics Involved in the Field of Auto Mechanics." Master's thesis, University of Chicago, 1925.

Hering, Carl. "Mathematics a Useful but Slippery Tool," *School Science and Mathematics*, XXVI (May, 1926), 467–75.

Hull, Evelyn. "A Project in the Social Application of Arithmetic," *Elementary School Journal*, XXIV (May, 1924), 673–78.

Karpinski, Louis C. "The Permanent Nature of the Necessity of Mathematics in the Secondary Schools," *Mathematics Teacher*, XXII (April, 1929), 197–202.

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Lindquist, T. "Application of Business Principles in Junior High School Mathematics," *School and Society*, XII (October, 9, 1920), 304–7.

Lovitt, W. V. "Continuity in Mathematics and Everyday Life," *Mathematics Teacher*, XVII (January, 1924), 31–34.

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Stratton, W. T. "Mathematics and Life," *School Science and Mathematics*, XV (February, 1915), 115–20.

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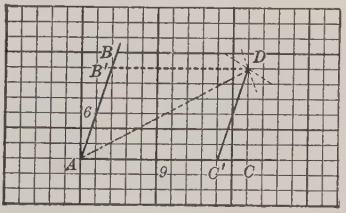
Motivating the study of mathematics by showing the need of mathematics in other school subjects.—Uses of mathematics in other school subjects make an exceptionally strong appeal because they are directly related to the pupil's experiences. The following illustrate the uses of geometry in physics. The first is an application of a geometric construction[1] and the second of one of the theorems on concurrent lines.[2]

1. "A boat is being rowed across a stream in the direction AB, 30° east of north at a rate of 6 miles an hour, while the current moves it east in the direction AC at the rate of 9 miles an hour. Find the direction in which the boat is actually moving.



"*Solution:* On squared paper lay off 9 units on AC, and 6 units on AB. Thus, $AC' = 9$ and $AB' = 6$.

"Construct with ruler and compasses the parallelogram $AC'DB'$. Draw the diagonal AD. Then the *direction* of the diagonal AD is the direction in which the boat is actually moving, and the *length* of AD is the number of miles the boat has passed over in 1 hour.

[1] *Ibid.*, p. 121. [2] *Ibid.*, p. 76.

"Since the sides AC' and AB' represent the velocity of the current and the rate of rowing, respectively, $AC'\ DB'$ is called the *parallelogram of velocities*. AC' and AB' are the *component velocities*. The diagonal AD is the *resultant velocity*."

2. "From cardboard cut a triangle. Draw the three medians of the triangle. If the construction is made carefully, the three medians will meet at a point. If the triangle is supported by placing a pin under the point of intersection, the triangle will be found to balance. For this reason the point of intersection of the three medians of a triangle is called the *center of gravity* of the triangle. It will be proved that the three medians of a triangle always meet in a point."

Literature relating to the uses in other school subjects.—The following references deal with the uses in school subjects other than mathematics.

Barton, S. G. "The Uses for Mathematics," *Science*, XL (November 13, 1914), 697–700.

Collins, J. V. "Calculations by Geometry of Astronomical Distances," *School Science and Mathematics*, XX (May, 1920), 416–18.

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Reagan, G. W. "Mathematics Involved in Solving High School Physics Problems," *School Science and Mathematics*, XXV (March, 1925), 292–99.

Tyler, H. W. "Mathematics in Science," *Mathematics Teacher*, XXI (May, 1928), 273–79.

Arousing interest by means of historical references.—Historical and biographical notes appeal strongly to certain pupils and add to the interest of all. Writers of textbooks have long ago recognized this. Some insert the historical material in the form of footnotes; others present it in the body of the text. Talks by teachers, or by pupils who have developed special interest in the history of the subject, add much to the general interest. It is decidedly worth while to let the pupils see how closely the development of civiliza-

tion is linked to that of mathematics. The following example shows how historical material may be used for the purpose of motivation.[1]

How standard units of length have been made.—Most of the civilized nations have derived a unit of length from the length of the human foot. The result has been that standard units of length are not the same in different countries.

Early units of measurements were indefinite. Thus the *foot-breadth* and *cubit* (distance from the elbow to the extremity of the middle finger) mentioned in the Bible vary for different persons, and exact measurement with these units is impossible. Hence it became necessary for people to adopt a definite unit.

In a book on surveying we find the following account of how people in the sixteenth century tried to obtain a standard unit. "Stand at the door of a church on Sunday, and bid sixteen men to stop, tall ones and small ones, as they happen to pass out when service is finished; then make them put their left feet one behind the other and the length thus obtained shall be a right and lawful rod to measure and survey land with, and the sixteenth part of it shall be a right and lawful foot."

Henry I, King of England (1100–1135 A.D.), is said to have used as a unit the distance from the point of his nose to the end of his thumb, approximately a yard's length. Other units were made in 1490 by Henry VIII and in 1588 by Queen Elizabeth. The present English standard was adopted in 1855.

In 1856 the English Government sent to this country two copies of the new English standard, one made of bronze, the other of iron, which were used as standards until 1875. The table below gives the units of linear measure in that system.

TABLE OF LINEAR MEASURE IN THE ENGLISH SYSTEM

12 inches (in.)	=1 foot (ft.)
3 feet (ft.)	=1 yard (yd.)
5½ yards = 16½ feet	=1 rod (rd.)
320 rods = 5,280 feet	=1 mile (mi.)
6 feet	=1 fathom
1.151 miles an hour	=1 knot

During the French Revolution the National Assembly (1790) appointed a committee of the Academy of Sciences to study the matter of finding a suitable system of weights and measures. This commission selected as the standard unit of length one ten-millionth part of the distance from

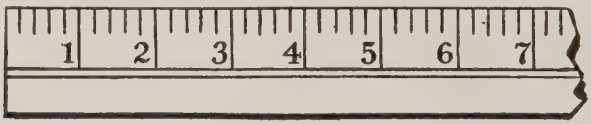
[1] E. R. Breslich, *Seventh-Year Mathematics* (New York: Macmillan Co., 1929), pp. 58–61.

the north pole to the equator, measured along the meridian through Paris. This standard unit is called *meter*. The commission determined the length of a meter to be about 39.37 inches, the work requiring seven years for its completion. This distance was marked off by expert instrument makers on a bar of platinum. On June 22, 1799, the standard unit was presented to the Council of Five Hundred and deposited in the archives at Paris. It is the only unit of length which is the result of scientific investigation. The meter (m.) is divided into 10 equal parts called *decimeters* (dm.). Each decimeter is divided into 10 equal parts called *centimeters* (cm.). Each centimeter is divided into 10 equal parts called *millimeters* (mm.). A thousand meters make a *kilometer* (km.). The following table expresses these lengths in terms of inches, feet, yards, and miles.

TABLE OF LINEAR MEASURE IN THE METRIC SYSTEM

	Inches	Feet	Yards	Miles
Millimeter	0.03937	0.003	0.001	
Centimeter	0.3937	0.033	0.011	
Decimeter	3.937	0.328	0.109	
Meter	39.37	3.281	1.093	
Kilometer	39,370.	3,280.833	1,093.611	0.621

We shall speak of this system of measures as the *metric system*. It is now generally used throughout the world in scientific investigations. In view of the trade between nations it is desirable that they employ a uniform system of measures in commerce. In 1866 a law was passed making the metric system legal in the United States. The system is easier to understand and to use than our common system. The figure below represents a part of a ruler graduated according to the metric system.



It should be kept in mind by the teacher that one of the major purposes of the historical material of a course is to arouse interest. Hence, it should not be assigned as homework to be recited the next day, and it should not be used as test material. This might easily change into distasteful drudgery a task intended for pleasure.

Literature relating to the use of history in teaching mathematics.— A careful reading of the references below will reveal many sugges-

tions as to the use of historical material in the teaching of mathematics.

Andrews, Frances E. "The Romance of Logarithms," *School Science and Mathematics*, XXVIII (February, 1928), 121–30.

Betz, William. "The Origin of Mathematics," *Mathematics Teacher*, XV (May, 1922), 283–93.

Breslich, E. R. "Arithmetic One Hundred Years Ago," *Elementary School Journal*, XXV (May, 1925), 664–74.

Burgess, E. G. "Mathematics," *School Science and Mathematics*, XXIV (March, 1924), 264–72.

Cajori, Florian. *History of Mathematics*. New York: Macmillan Co., 1914.

Fergusen, Zoe. "A Thread of Mathematical History and Some Lessons," *School Science and Mathematics*, XXIV (January, 1924), 37–45.

Hassler, J. O. "The Use of Mathematical History in Teaching," *Mathematics Teacher*, XXII (March, 1929), 166–71.

Karpinski, L. C. "The Parallel Development of Mathematical Ideas, Numerically and Geometrically," *School Science and Mathematics*, XX (December, 1920), 821–28.

Karpinski, L. C., and Fiedler, A. M. "The Terminology of Elementary Geometry," *ibid.*, XXIV (February, 1924), 162–67.

Kimmel, H. "The Status of Mathematics and Mathematical Instruction during the Colonial Period," *School and Society*, IX (February 15, 1919), 194–202.

Miller, G. A. "Mathematical Shortcomings of the Greeks," *School Science and Mathematics*, XXIV (March, 1924), 284–87.

———. "The Development of the Graph for Expressing Functionality," *ibid.*, XXVIII (November, 1928), 829–34.

———. "Modern Developments in Elementary and Secondary Mathematics," *ibid.*, XVII (January, 1917), 32–42.

———. "The Development of the Function Concept," *ibid.*, XXVIII (May, 1928), 506–16.

Simons, Lao G. "The Place of the History and Recreation of Mathematics in Teaching Algebra and Geometry," *Mathematics Teacher*, XVI (February, 1923), 94–101.

Smith, D. E. *History of Mathematics*. New York: Ginn & Co., 1923.

Vallandingham, J. T. "The Value and Method of the Historical Element in the Teaching of Secondary Mathematics," *School Science and Mathematics*, XXI (December, 1921), 817–22.

Growth in mathematical power stimulates interest in the subject.— The fact that a student of mathematics constantly improves his

methods and increases his power to do work more effectively and with decreasing effort is a strong incentive to continue the study of mathematics. It should be shown how new methods replace others that the pupil has outgrown. The problem of finding an unknown distance by indirect measurement is a good illustration. At first the congruent triangle method is used.[1] This is replaced by the scale-drawing method, which in turn gives way to the similar triangle method. Finally the pupil solves the problem by the method that is actually used by the surveyor, i.e., by the trigonometric method. He sees that, as he passes from each method to the next, he is making steady improvement. He learns how mathematics helps him to save time and work, and at the same time increases the accuracy of his results. The prospect of learning to do a piece of work with less effort and better results appeals to him. In fact, one of the most powerful arguments for the continued study of the subject is that some things can be done better by geometry than by arithmetic; again better by algebra than by arithmetic or geometry; and still better by trigonometry than by arithmetic, geometry, or algebra. Each course points the way to the one which is to follow.

The desire to extend useful knowledge motivates the study of mathematics.—Experiences in one unit suggest similar experiences with another. When the pupil who has learned to appreciate the value of performing the operations with literal numbers comes to the study of negative numbers, he often raises the question whether negative numbers can be added, subtracted, multiplied, and divided according to the laws developed for literal numbers. The answer to the question is a natural introduction to the unit on the operations with positive and negative numbers. The fact that his knowledge of solving linear equations enabled him to solve problems leading to such equations is a strong motive for a thorough study of quadratic equations as soon as the first problem is encountered that leads to a quadratic equation.

The logical motive.—The value of practical applications as a motive for the study of mathematics is generally conceded. However, many teachers feel that the logical motive is even more powerful than the practical, which they regard as commonplace. They argue that mathematics would never have secured a place on the

[1] E. R. Breslich, *Eighth-Year Mathematics*, pp. 180–240.

curriculum merely because it is practical in adult life, that their best pupils have but slight interest in the subject because it is useful in the activities of older people, and that the principal value of the study of mathematics lies in the training it offers in exact and well-formulated thinking and in the acquisition of certain modes of thought which everyone should develop.

The logical motive must not be slighted. Its influence should be felt in the teaching of mathematics. The pupil must be led to appreciate the value to be derived from constant practice in the type of reasoning found in mathematics. The real stimuli for the study of the subject are intellectual growth, success, self-activity, personal pleasure, and understanding.

Literature relating to the cultural values of mathematics.—The following reading list contains many suggestions as to the cultural values of mathematics.

Carmichael, R. D. "The Larger Human Worth of Mathematics," *Scientific Monthly*, XIV (May, 1922), 447–68.

Dresden, A. "Why Study Mathematics?" *School and Society*, XII (October 30, 1920), 390–95.

Georges, J. S. "Functional Relations and Mathematical Training," *School Science and Mathematics*, XXVI (October, 1926), 689–704.

Hawkes, H. E. "Educational Values in Mathematical Training," *Mathematics Teacher*, IV (December, 1911), 75–78; V (March, 1912), 85–89.

Hill, Frank A. "The Educational Value of Mathematics," *Educational Review*, IX (April, 1895), 349–58.

Kane, Sr. M. Gabriel. "The Cultural Value of Mathematics," *Mathematics Teacher*, XV (April, 1922), 228–36.

Karpinski, L. C. "The Methods and Aims of Mathematical Science," *School Science and Mathematics*, XXII (November, 1922), 718–22.

Kelly, T. L. "Values in High School Algebra and Their Measurement," *Teachers College Record*, XXI (May, 1920), 246–90.

Kempner, Aubrey J. "The Cultural Value of Mathematics," *Mathematics Teacher*, XXII (March, 1929), 127–45.

Keyser, C. J. "The Human Worth of Rigorous Thinking," *Teachers College Record*, May, 1917.

Lennes, N. J. "Mathematics for Culture," *Educational Review*, XLVII (May, 1914), 469–77.

Lovitt, W. V. "Continuity in Mathematics and Everyday Life," *Mathematics Teacher*, XVII (January, 1924), 31–34.

———. "Imagination in Mathematics," *ibid.*, XVII (May, 1924), 263–69.

McLaughlin, Henry P. "Geometry as a Course in Reasoning," *ibid.*, XVI (December, 1923), 493–500.

Martin, W. "Mathematical Training as an Aid in Removing Limitations," *School Science and Mathematics*, XXIII (April, 1923), 347–52.

Mayo, C. H. P. "The Position of Mathematics," *Educational Review*, LVII (March, 1919), 194–204.

Minnick, John H. "The Cultural Value of Secondary Mathematics," *Mathematics Teacher*, XVI (January, 1923), 35–40.

Moore, C. N. "On the Disciplinary and Applied Values of Mathematical Study," *Education*, XXXIX (December, 1918), 209–16.

———. "Mathematics in General Education," *School and Society*, IX (January 4, 1919), 31–33.

Moore, E. C. "Does the Study of Mathematics Train the Mind Specifically or Universally?" *ibid.*, VII (June, 1918), 754–64.

Newhall, C. W. "The Real Problem in Secondary Mathematics," *Educational Review*, XLIII (May, 1912), 472–82.

Perry, Winona. "What Are the Real Values of Geometry?" *Mathematics Teacher*, XXI (January, 1928), 51–55.

Rankin, W. W. "The Cultural Value of Mathematics," *ibid.*, XXII (April, 1929), 215–23.

Schreiber, Edwin W. "Some Ideals in Teaching Mathematics," *ibid.*, XIV (May, 1921), 252–55.

Shaw, J. B. "A Chapter on the Aesthetics of the Quadratic," *ibid.*, XXI (March, 1928), 121–34.

Slaught, H. E. "Mathematics and Sunshine," *ibid.*, XXI (May, 1928), 245–52.

———. "Romance of Mathematics," *ibid.*, XX (October, 1927), 303–9.

Smith, D. E. "Aesthetics and Mathematics," *ibid.*, XX (December, 1927). 419–28.

———. "Religio Mathematici," *ibid.*, XIV (December, 1921), 413–27.

Stroup, P. "Direct Cultural Motivation for Demonstrative Geometry," *ibid.*, XX (March, 1927), 173–77.

Maintaining interest in the study of mathematics.—It has been argued in the foregoing pages that the pupil's interest in mathematics should be aroused by showing the logical motives for the study and the practical values to be derived from the work in mathematics. It is not to be inferred that a motive must be supplied for everything that is to be taught. It is desirable, however, to have the learner feel that the subject is decidedly worth while and that not to study mathematics would be a real personal loss. It

is therefore important that interest once aroused be maintained. Fortunately, there is no scarcity of quantity or variety of useful and practical material that can be understood, appreciated, and enjoyed by pupils of the secondary-school period. It is only necessary to call attention to the geometric forms and mathematical relationships that are found in their environment and experiences. The facts and principles of intuitive geometry are most easily related to the world and life outside of the schoolroom, but also in the later logical geometry much is gained by correlating abstract principles established by logic with practical applications.

However, the resourceful teacher has no difficulty in discovering other devices for arousing and maintaining interest. Mathematical games and puzzles, fallacies and optical illusions, contests and plays, exhibits, club and assembly programs, have been of greatest value in reaching individuals, groups, or even the school as a whole. The teacher will find ample information about the uses of the foregoing devices in the bibliography given at the end of this chapter and in chapter iv. Much suitable material can also be gathered from mathematics textbooks.

Mathematical recreations.—Pupils, as well as adults, find much pleasure in attempting to solve the mysteries of mathematical puzzles, fallacies, and illusions. Such material, properly related to class work, can, therefore, be made a valuable aid to teaching. The following examples are good illustrations.

Example 1: To prove that any number is equal to any other number.—This example may be introduced when a pupil attempts to divide a number by zero. If we admit zero as a divisor, all numbers can be shown to be equal. While the teacher presents what seems to be a proof, the pupils are expected to follow carefully and to locate the error.

Let a and b be any two given numbers. Let c and d be two equal numbers.

$$\begin{array}{ll} \text{Then} & c=d \\ \therefore & ac=ad \\ \text{and} & bc=bd \end{array}$$

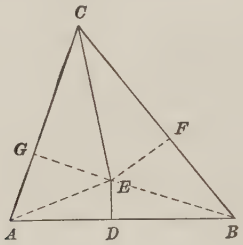
$$\begin{array}{l} \therefore\ ac-bc=ad-bd \\ \therefore\ ac-ad=bc-bd \\ \therefore\ a(c-d)=b(c-d) \\ \therefore\ a=b \end{array}$$

Example 2:[1] *To prove that every triangle is isosceles.*—This example might be given when it becomes necessary to impress pupils with the importance of careful thinking and observing in a logical demonstration. The pupil is to discover the fallacy in the proof.

"*Given* any triangle, as ABC.

To prove that ABC is isosceles.

Proof: Let DE be the perpendicular bisector of AB, and let CE be the bisector of angle C, meeting DE at E.



From E draw EA and EB.

Draw EG perpendicular to AC, and EF perpendicular to CB.

Then $\triangle ADE \cong \triangle BDE$.

Hence, $AE = BE$.

(Since corresponding sides of congruent triangles are equal.)

$\triangle CEG \cong \triangle CEF$.

Hence, $EG = EF$ and $CG = CF$.

Therefore, $\triangle AEG \cong \triangle BEF$.

Hence, $GA = FB$.

Since $CG = CF$,

it follows that

$$CG + GA = CF + FB,$$

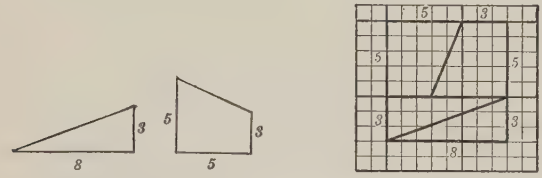
or $$CA = CB.$$

Therefore the triangle ABC is isosceles."

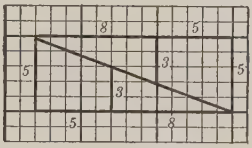
Example 3: To show that 64 is equal to 65.—"Draw two right triangles like the one shown below, having the sides, including the right angle, equal to 3 and 8, respectively. Draw two quadrilaterals like the one shown be-

[1] E. R. Breslich, *Senior Mathematics*, Book II, pp. 45, 46.

low, having one pair of opposite sides parallel and equal to 3 and 5, respectively, and the third side perpendicular to the parallel sides and equal to 5.



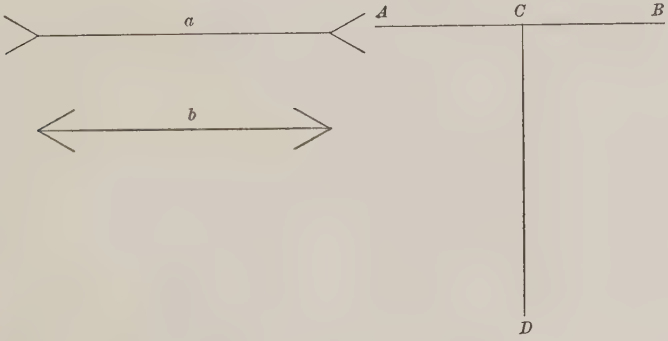
"Form a square by placing the triangles and quadrilaterals as shown. The area of this square is equal to $8\times8=64$.



Now place the four figures so as to form a rectangle. The area of the rectangle is equal to $13\times5=65$.

Hence, $64=65$."

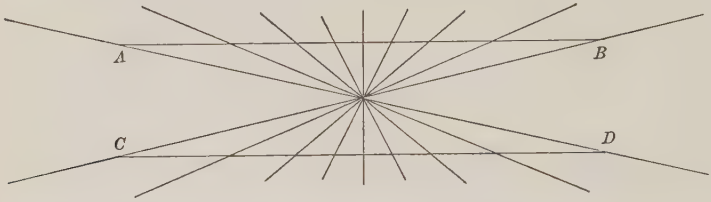
Example 4:[1] *The purpose of this example is to show the danger of depending too much on the drawing in working out a proof.*—"Relations which appear to the eye to be correct may be erroneous and misleading. For



example, the two line segments a and b, and AB and CD, in the figures above, appear to be of unequal lengths, but careful measure-

[1] *Ibid.*, p. 2.

ments show that they are equal. The two lines *AB* and *CD*, below, appear to be curved, but when they are tested with a straightedge it is seen that they are straight. As long as mathematicians failed to develop methods better than observation, their achievements had to be very limited."



Literature relating to mathematical recreations.—The teacher will find numerous suggestions on mathematical recreations in the following articles and books.

Booth, A. L. "A Mathematical Recreation,"—"Some Angles of a Right Triangle," *Mathematics Teacher*, XI (June, 1919), 177–81.

Colwell, R. C. "A Rule To Square Numbers Mentally," *School Science and Mathematics*, XIV (November, 1921), 71–77.

Karpinski, L. C. "A Rule To Square Numbers Mentally," *ibid.*, XV (January, 1915), 20–21.

McLaughlin, H. P. "Algebraic Magic Squares," *Mathematics Teacher*, XIV (February, 1921), 71–77.

Newhall, C. W. "Recreations in Secondary Mathematics," *School Science and Mathematics*, XV (April, 1915), 277–93.

Raster, Alfreda. "Mathematical Games," *Mathematics Teacher*, XVII (November, 1924), 422–25.

Sanford, Vera. "Magic Circles," *ibid.*, XVI (May, 1923), 348–50.

Ball, W. W. R. *Mathematical Recreations and Essays*. New York: Macmillan Co., 1914.

Jones, S. I. *Mathematical Wrinkles*. Nashville, Tennessee: published by S. I. Jones, 1923.

Shubert, H. *Mathematical Essays and Recreations*. Chicago: Open Court Publishing Co., 1918.

Weeks, Raymond. *The Boys' Own Arithmetic*. New York: E. P. Dutton & Co., 1924.

White, F. E. *Scrapbook of Elementary Mathematics*. Chicago: Open Court Publishing Co., 1910.

Mathematics clubs.—When a department succeeds in arousing interest in mathematics, it is often advisable to form an organiza-

tion of the pupils who have developed special interests. The purpose is to be able to carry on lengthy or detailed discussions which the limited time of the class period must necessarily exclude. The literature at the end of this chapter contains a wealth of suggestions as to method of organizing and conducting mathematics clubs. There is no scarcity of suitable material for the programs. Historical topics relating to the various mathematical subjects, problems arising in the classroom which require further study and discussion, reports on voluntary projects carried to completion by individual pupils, exhibits of homemade mathematical instruments, famous mathematical problems, biographies of famous mathematicians, and an abundance of other material are easily combined into profitable programs. Occasionally a social program should be inserted to establish closer personal relations among pupils and teachers.

Assembly programs.—Many schools make it a rule to have the various departments furnish some of the programs for the school assemblies. This is an excellent opportunity for a department to arouse the interest of the whole school. The program might be given by the mathematics club, by pupils selected from various classes, or by a single class. Mathematical contests or plays are very effective. The writer saw the program on page 84 staged successfully by a junior high school class in Chicago.

Each number, or act, on the program was carried out by one or more pupils on the stage of the assembly room, in the presence of the whole school. The acts were connected with each other in the form of a story telling the historical development of the number system.

Summary.—The chapter has shown how interest in mathematics may be aroused and maintained by motivating the study. The following motives appeal strongly to the secondary-school pupils:

1. The social uses of the subject
2. The practical uses
3. The uses of mathematics in other school subjects
4. Discussions of the history of the development of mathematics
5. Power gained through the study of mathematics
6. The cultural values

The following devices are excellent to arouse interest in mathematics:

1. Mathematical recreations and puzzles
2. Mathematics clubs
3. Mathematical assembly programs
4. Mathematical plays

MANLEY
JUNIOR HIGH SCHOOL

Chicago
October 16, 1929

Program

By Room 112, Arithmetic

Miss Kuebler

I. Mathematical Recreations
   (A) Number stories of long ago
      1. How Ahmes and Chang wrote their numbers
      2. How Hippias, Daniel and Titus wrote their numbers
      3. How Gupta and Mohammed wrote their numbers
      4. How Cains added
         a. The Roman abacus
         b. The Greek abacus
         c. The Chinese abacus
         d. The Modern Adding Machine
      5. Leonardo of Pisa
      6. Multiplication 500 years ago
      7. How Filippo divided years ago
      8. How Filippo proved his work
      9. Division 300 years ago
      10. How Jacob added fractions 400 years ago
      11. How Simon wrote decimal fractions 400 years ago

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Anning, Norman. "Socrates Teaches Mathematics," *School Science and Mathematics*, XXIII (June, 1923), 581–84.

Anonymous. "Flatland, A Mathematics Play," *ibid.*, XIV (October, 1914), 583–87.

Boughn, E. T. "A Mathematical Contest," *ibid.*, XVII (April, 1917), 329–30.

Crawford, Alma E. "A Little Journey to the Land of Mathematics," *Mathematics Teacher*, XVII (October, 1924), 336–42.

Graff, Margaret. "Euclid Dramatized," *School Science and Mathematics*, XXI (April, 1921), 381–83.

Hansen, Lena, B. "Creating Interest in Mathematics through Special Topics," *Mathematics Teacher*, XXIII (January, 1930), 3–7.

Harding, P. H., and Mildred Travis, Jeannette Bower, Katherine Hegeman, and Stella Chamberlen. "A Mathematical Victory," *School Science and Mathematics*, XVII (June, 1917), 475–82.

Harris, Isabel. "A Geometric Recreation," *ibid.*, XX (November, 1920), 731–33.

Koch, E. H., Jr. "Mathematics Contests," *Mathematics Teacher*, IX (June, 1917), 179–86.

Rorer, J. T. "A New Form of School Contest," *Educational Review*, LVII (April, 1919), 339–45.

Schlierholz, T. "A Number Play in Three Acts," *Mathematics Teacher*, XVII (March, 1924), 154–69.

Schorling, Raleigh. "A Mathematical Contest," *School Science and Mathematics*, XV (December, 1915), 794–97.

Whitaker, Helen. "The Math Quest," *ibid.*, XX (May, 1920), 457–67.

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Brown, I. M. "A Mathematical Club in a Girls' School," *Journal of Education*, XXXVIII (September, 1916), 556.

Gegenheimer, Frank C. "Mathematics Clubs," *School Science and Mathematics*, XVI (December, 1916), 791–92.

Gugle, Marie, and Others. "Recreational Values Achieved through Mathematics Clubs in Secondary Schools," *First Yearbook, The National Council of Teachers of Mathematics* (1926), pp. 194–200.

Gulden, Minerva. "Mathematics Club Program," *Mathematics Teacher*, XVII (October, 1924), 350–58.

Hatcher, Frances B. "A Living Theorem," *School Science and Mathematics*, XVI (January, 1916), 39–40.

Hatton, Mary C. "A Mathematics Club," *Mathematics Teacher*, XX (January, 1927), 39–45.

McSorley, Kathryn. "Mock Trial of B versus A," *School Science and Mathematics*, XVIII (October, 1918), 611–21.

Miller, Florence, Brooks. "A Near Tragedy," *Mathematics Teacher*, XXII (December, 1929) 472–81.

Newhall, Charles. "A Secondary School Mathematics Club," *ibid.*, XI (June, 1911), 500–509.

Phillips, K. G., "Junior High School Mathematics Club," *High School Journal*, XIII (February, 1930), 68–71.

Reed, Zulu. "High School Mathematics Clubs," *Mathematics Teacher*, XVIII (October, 1925), 341–63.

Russell, Helen. "Mathematics Clubs," *ibid.*, XVII (May, 1924), 283–85.

Shoesmith, B. I. "Mathematics Clubs in Secondary Schools," *School Science and Mathematics*, XVI (February, 1916), 106–13.

Slaught, H. E. "The Evolution of Numbers—An Historical Drama in Two Acts," *Mathematics Teacher*, XXI (November, 1928), 305–15.

Snell, C. A. "Mathematics Clubs in High School," *ibid.*, VIII (December, 1915), 73–78.

Snyder, Ruth, L. "If," A Play in Two Acts, *ibid.*, XXII (December, 1929), 482–86.

Steward, Margaret. "The Mathematics Club at the Pontiac High School," *ibid.*, XXIII (January, 1930), 25–29.

Taylor, Helen, "The Mathematical Library and Recreational Programs," *School Science and Mathematics*, XXX (June, 1930), 626–34.

Webster, Louisa M. "Mathematics Club," *Mathematics Teacher*, IX (June, 1917), 203–8.

Wheeler, A. H. "Mathematics Club Program," *ibid.*, XVI (November, 1923), 385–90.

Whitaker, Helen. "The Math Quest," *School Science and Mathematics*, XX (May, 1920), 457–59.

Young, J. W. A. *Teaching of Mathematics*, pp. 163–69. New York: Longmans, Green & Co., 1911.

CHAPTER IV

TEACHING PUPILS HOW TO STUDY MATHEMATICS

Study habits are important factors in success in mathematics.—Success in a high-school subject depends upon a variety of factors. The most outstanding determining factors are, of course, the pupil's general and special abilities; but former experiences, industry, age, and interest are very important. Furthermore, careful diagnosis has often shown that when a pupil has difficulty in mathematics the cause may not be the subject matter which he is trying to master but the lack of certain study habits which he has failed to acquire. The following incident illustrates this. A class in solid geometry was asked to study a page in the textbook which developed the formula for finding the lateral area of a circular cone. When reports were called for, the first pupil failed completely. He said he was unable to understand any of the assignment. The second pupil made a good start with his report but was soon lost and could not proceed. The third pupil gave a very good report. Asked if they understood his discussion, the other two said that they understood it perfectly. The teacher now asked the first pupil to tell why he was previously unable to understand the work. His self-analysis revealed that he actually understood everything until he came to the word "sector." He did not know the meaning of that word and did not go farther. A similar analysis by the second pupil revealed that he was unable to finish the assignment because he did not remember the formula for finding the area of a sector. The third pupil said that he also did not know the formula, but that he had turned to the list of formulas given in the back of the book, and that, having found the formula he needed, he was able to proceed.

Thus, the first two pupils had really no difficulty with the mathematical subject matter, for they understood it perfectly when it was presented by the third pupil. However, they did not succeed because they did not have the proper study habits to carry them through. One lacked the habit of using the index; the other failed to use the tables and formulas. Both knew that these devices were

provided in the book, but they were not in the habit of using them.

Careful observation of pupils at study and detailed analyses of test material verify the contention that one of the greatest single sources of difficulty in mathematics is the lack of good study habits, and not the subject matter or inability of the pupil to understand it. "One becomes most acutely aware of the necessity of improving methods of study when a pupil is in difficulty. It is evident in such a case that the pupils' methods are inadequate."[1] If this is true, teachers must learn to give instruction in ways of acquiring habits of study that is fully as effective as instruction in acquiring subject matter. The fact is that teachers as a rule are not sufficiently trained in psychology to direct the pupils in their efforts to improve their methods of study. "They can ask questions in their subjects, but they do not know about pupils' minds in a way which makes it possible to tell pupils how to study."[2]

In his chapter on "Learning and Its Supervision"[3] Judd has suggested a number of procedures which a teacher may adopt with a class that he meets for the purpose of discussing the problem of how to study. The teacher who wishes to learn how to deal effectively with the study habits of pupils will find the reading of that chapter most profitable. Some of the important habits of study in which pupils must receive instruction are to be discussed in the following pages.

The need of teaching high-school pupils to study.—In the past the schools have left it largely to the learner independently to develop proper study habits. It has been assumed that intellectual independence is attained eventually by all pupils if, day after day during the entire high-school period, they are required to study at home the lessons assigned by the teachers. This view has been firmly established in the minds of many teachers and administrators and has gone unchallenged for years. In fact, only recently has serious attention been given to ways of teaching pupils to study, and the problem is still far from solution.

It is futile to assign lessons in mathematics to a pupil who does not know how to study and is therefore unable to do the work. Fail-

[1] C. H. Judd, *Psychology of Education*, p. 508.

[2] *Ibid.*, p. 508. [3] *Ibid.*, pp. 509–36.

ing to succeed day after day, and dreading to face the teacher again and again without the completed assigned tasks, some become discouraged or indifferent, others secure help at home or elsewhere, and still others submit as their own the work done by someone else. These practices are so common among pupils that they are not even considered dishonest, and pupils have come to believe that there exists a double standard of honesty. At any rate they consider dishonesty entirely legitimate under certain conditions prevailing in schools.

The following story, which appeared some years ago in the *Ladies' Home Journal* (January, 1913), illustrates what is taking place in many homes every day. It shows that much of the assigned home work is not done by the pupils.

A widow came to the superintendent of schools with the following complaint:

> I have four little girls attending your schools. I am up at five o'clock in the morning to get them off to school and to get myself off to work. It is six o'clock in the evening when I reach home again, pretty well worn out, and after we have had dinner and have tidied up the house a bit, it is eight o'clock. Then, tired as I am, I sit down and teach the little girls the lessons your teachers will hear them say over on the following day. Now, if it is all the same to you, it would be a great help and favor to me if you will have your teachers teach the lessons during the day, and then all I would have to do at night would be to hear them say them over.

The story illustrates that pupils do not study independently at home. Not all parents are educators; and the help they give only too frequently amounts to doing the work for the children, which, of course, is of little or no educational value.

There is statistical evidence to show that many children do not do the assigned home work and that they do not develop study habits. For example, Reavis[1] carried on an investigation involving the children coming from 393 homes. He divided the homes into three equal groups which were ranked as I, II, and III according to moral atmosphere, the educational interest of the parents, and the extent to which the parents provided for food, clothing, health, entertainment, and literature. Likewise, the pupils were classified

[1] W. C. Reavis, "Factors That Determine Habits of Study," *Elementary School Journal*, XII (October, 1911), 71–81.

in three groups according to their habits of study. The results are summarized in the following table.

Students Having Habits of Study of Quality	Percentage of Students from Homes I	Percentage of Students from Homes II	Percentage of Students from Homes III
I	75.0	32.4	15.3
II	19.7	48.2	40.7
III	5.3	19.4	44.0

Read: Seventy-five per cent of pupils from homes of quality I had study habits of quality I, etc.; and only 15.3 per cent of pupils from homes of quality III had study habits of first quality.

The table indicates that study habits are conditioned by home environment. If home work alone is depended upon, students from the less favored homes stand a smaller chance to cultivate habits of study than their more fortunate companions, and this phase of their development is going to be neglected.

Reavis shows further that many students habitually neglect the preparation at home, particularly those who come from the less favored homes:

Rank of Homes	Distribution of Students	
	Doing Home Study	Not Doing Home Study
I	38.5	4.1
II	54.2	43.8
III	7.3	52.1

Read: Of the pupils doing home study, 38.5 per cent came from homes of quality I, etc. Only 4.1 per cent of pupils not doing home study came from homes of quality I.

Apparently the pupils who are most in need of home study are likely to be the ones who fail to do it on account of unsatisfactory home surroundings. It is assumed that as a rule the better pupils learn to study without much assistance. "Yet," says Judd, "the more important problem is to make increasingly efficient those pupils who are not in difficulty. Some teachers do not realize that bright pupils need help in methods of work."[1]

High-school graduates do not know how to study.—The most common method of instruction in colleges has been the so-called "home-work-recitation plan." Lessons are assigned to be mastered outside

[1] C. H. Judd, *op. cit.*, p. 508.

of the classroom, and recitations are held the following day. It is the plan with which college graduates are most familiar; and when they take up teaching, they naturally use it with the high-school pupils in their classes. Indeed, one of the principal arguments to support the plan in the high school is its almost exclusive use in the colleges. It seems to be taken for granted that a different plan would be fatal to the success in college work as it is most commonly conducted.

However, when we listen to the frequent complaints of college teachers that high schools fail to give training in proper study habits, we begin to question the validity of that assumption. As a matter of fact, ever so many college Freshmen do not know how to study. This statement is readily verified by the testimony of the college students themselves. They frankly admit that it takes the ordinary college Freshman months to learn how to study. Moreover, college teachers who take a genuine interest in the success of their students are aware of this fact and give them definite instructions as to studying their lessons. For example, in his textbook on college algebra, Professor Wilczynski[1] offers a number of study hints for college students. Among others, he gives such basic study helps as the following:

Carefully read the assigned lesson. Before turning to the examples, convince yourself that you are really able to follow the logic of the argument. . . . . Try to discover a fundamental idea which runs through the argument and illustrates it. . . . . See whether you can appreciate the practical importance of the subject under discussion. . . . . If your self-examination has had an unsatisfactory result, find out how to classify your difficulty. . . . . Find out where the argument begins to become unintelligible for the first time and fix your attention on the statement made at this point. . . . . Search your memory and your text for passages dealing with the subjects that caused the difficulty. Use the index and table of contents. . . . . Often you will be able to answer your own question after having put it in written form. . . . . Make your oral and written statements clear and unambiguous. As a test of clearness, imagine yourself addressing a person of intelligence who is properly prepared to follow your argument, but do not expect him to do any mind-reading. . . . . Good paper, good ink, and neatness in form are of great importance in orderly thinking. . . . . Some students rush to the examples before reading the text. You will fail

[1] E. J. Wilczynski, *College Algebra* (Allyn and Bacon), pp. 9–10.

if you adopt this plan. Even if you should succeed in solving the examples, you will probably fail to understand the principles which they are intended to illustrate. . . . .

Evidently the author of this college algebra textbook has a keen understanding of his students and has discovered that the lack of certain study habits is responsible for a great deal of the difficulty they have in college mathematics. He has definitely determined what the important study habits are and proceeds to give instruction in the most elementary study habits, which are commonly assumed to be developed in the high school. It is apparent that the secondary schools have not solved the problem to the satisfaction of college teachers.

The department of mathematics of Brown University[1] has had a similar experience. Not long ago the department published a circular containing suggestions as to methods of studying mathematics. All of these suggestions are so fundamental to successful studying in high school that we are amazed that attention should have to be called to them in college courses. The following selections are taken from this circular:

1. *Concentrate on your work during your time for study.* The power of focusing attention for an extended period of time is one of the most valuable things that a college can contribute, and this habit will do much to assure your success either in college or in the world.

2. *Do not allow avoidable interruptions by yourself or others.* A little will power enables a student to ignore minor disturbances and distractions. Education is a part of your business in life, and in the acquiring of it you should form business habits.

3. *Feel sure of success.* Few intelligent and industrious students are unable to comprehend fundamental processes of mathematical reasoning. If you are discouraged, consult with an instructor as to the cause of your difficulties and the remedy for your lack of success.

4. *Reflect as you read.* Learning by rote contributes little to educational development, and the habitual performance of tasks in a mechanical way leads to intellectual atrophy.

5. *Be mentally alert, active, and aggressive.* Apply the principles of the lessons to concrete problems; such application is the test of understanding and requires effort and initiative.

[1] Brown University, "Suggestions for Students of Mathematics and Life Activities," *Bulletin of the Department of Mathematics, May, 1921* (Providence, Rhode Island).

6. *Enjoy overcoming obstacles.* It is worth while in itself, gives power for surmounting new obstacles, and is good preparation for success in life. Love of difficulty is essential to high attainment.

7. *Work alone at least part of the time.* Discuss the subject with fellow students, but think it over and do detailed work by yourself both before and after such discussion.

8. *Attend class regularly.* Prepare each lesson regularly and systematically. In the logical development of the subject each lesson plays its part, so that a lack of understanding of earlier work is a bar to progress.

The preceding examples show that college teachers find that many high-school graduates do not possess the most fundamental study habits, and that they enter college with some very undesirable habits which interfere with successful work.

That this experience is general was shown by the widespread interest among colleges that was aroused by the Brown University circular. A similar circular for college Freshmen has been published by Beloit College.[1]

Ways of teaching high-school pupils how to study.—In connection with Reavis' investigation,[2] he reports a method of training pupils to study. He devised a program card which provided definite periods throughout the school day for class work, recreation, and study. Each pupil made up a program of study which he followed during the term. The card is as follows:

PUPILS' STUDY PROGRAM

Name:_____ Grade:_____

Hour	Study	Recite
9:00		
9:45		
10:30		
11:10		
1:15		
2:00		
2:45		
3:20		

[1] A. W. Burr, *Do You Know How to Study?* (Beloit, Wisconsin: Beloit College).

[2] W. C. Reavis, "Importance of a Study Program for High-School Pupils," *School Review*, XIX (June, 1911), 398–405.

The report states that three things have been accomplished with the cards: (1) The problem of discipline had been practically solved. (2) Considerable improvement had been made in scholarship. (3) A large majority of pupils had made provision for regular hours of home study. It is interesting to note the close relation existing between good study habits and school discipline.

Some high schools have attempted to solve the problem of teaching pupils to study by giving a printed list of study suggestions to all pupils. When the teacher finds a pupil who does not possess a certain habit of study, he is expected to call his attention to the particular suggestion in the list from which the pupil may derive help and benefit. The following study helps have been used successfully in the University of Chicago High School:

1. Form a time-and-place habit by studying the same subject in the same place, at the same time each day.

2. Have proper study conditions and equipment—a quiet room, not too warm; good light at the left; a straight chair and table; the necessary books, tools, and materials.

3. Study independently. Do your own work and use your own judgment, asking for help only when you cannot proceed without it, thus developing ability to think for yourself and the will power and self-reliance essential to success.

4. Arrange your tasks economically: study those requiring fresh attention, like reading, first; those in which concentration is easier, like written work, later.

5. Sit straight and go at the work vigorously, with confidence and determination, without lounging or waste of time. When actually tired, exercise a moment, open the window, change to a different type of work.

6. Be clear on the assignment and the form in which it is to be delivered. In class write the assignment down when it is made. Mark things to be carefully learned. When in doubt, consult the teacher.

7. In committing material to memory, learn it as a whole; go over it quickly first, then more carefully, and then again and again until you have it. In learning forms, rules, vocabularies, etc., it will help to repeat them aloud.

8. In studying material to be understood and digested but not memorized, first go over the whole quickly, then carefully section by section; if possible, then review the whole quickly.

9. Use judgment as well as memory; analyze paragraphs, select important points, note how minor ones are related to them; use your pencil

freely to mark important points so that you may learn systematically and review easily.

10. Study an advance assignment promptly and review before going to class; recall memorized matter by repeating it, aloud if necessary; think through a series of points to see that you have them in order in your mind.

11. Use all the material aids available—index, appendix, notes, vocabulary, maps, illustrations in your textbook, as well as other books and periodicals.

Lists similar to the foregoing have been in use in other high schools. They should be very helpful to the teacher who is seriously trying to help his pupils learn how to study. The difficulty with such rules and suggestions is that they are usually not accompanied by systematic applications in the everyday tasks of school work.

Some directions for developing study habits are given by Carter:[1]

Those pupils are early identified who have not developed study habits. As soon as the teacher recognizes the lack of a study habit, the pupil's attention can be called to it. By being shown the disadvantage under which he is laboring, he is brought to realize the value of the habit and the desirability of developing it. Under the careful direction of the teacher, he then begins to use it in his work.

An inventory test of the study habits of pupils.—The technique of teaching study habits should be worked out as carefully as that of teaching subject matter. The first step is to identify the study habits which the student actually possesses. This information might be obtained by observing each pupil carefully as he studies or by administering an inventory test on study habits. The latter is probably the quickest and best method. In making such a test, two procedures might be followed. One is to select occasionally an assignment which calls for certain habits of study and to request the pupils to write down each step they take as they do the assigned work. Another way of constructing the inventory test is to make a list of study habits and to ask questions about them. The following are typical test items:

Do you reflect as you read each sentence or do you read rapidly as you read a story?

[1] R. E. Carter, "Teaching a Study Habit," *School Review* XXIX (November and December, 1921), 695–706, 761–75.

Do you organize and review daily the work of the unit you are studying?

Do you recite to yourself or to someone at home in preparing yourself for reporting to the class?

When you come to a word or formula you do not know, do you look up its meaning?

Strictly mathematical study habits.—In addition to habits of study which are essential to success in every subject, others have to be established that are especially valuable in the study of mathematics. Since the advantages to the pupil possessing such habits are seen best in connection with certain mathematical situations, they will be discussed more fully later where and when the need of strictly mathematical habits becomes apparent. The following will suffice to illustrate what is meant:

1. The habit of expressing verbal statements in brief symbolic form, and of replacing large numbers by approximate values, thereby avoiding confusion and the resulting wasteful effort required in planning the solution of problems.

2. The habit of connecting words and symbols with meanings. Without this habit, pupils employ certain words and principles that are used day after day without ever attempting to comprehend what is meant. They continue to talk glibly in terms which they do not understand, and seem to be perfectly satisfied with the performance. If they are ever to outgrow this mechanical way of studying mathematics, they must form the habit of being sure of the meanings of the terms which they use in mathematical discussions. There are cases on record of high-school seniors who did not know the Roman numerals which designate the chapters in the textbook; of students of geometry who studied and discussed proportions for months without ever acquiring an understanding of the meaning of the term "proportion"; of students who finished a course in trigonometry but could not tell the meaning of the sine of a given angle. Such deficiencies are not explained by lack of ability. Very little can be done for these students until they improve their habits of study.

3. The habit of making a mental summary of what is known or given in a discussion or a problem and of what is to be proved or found. Both sets of facts must always stand out clearly in the mind

of the pupil whether he solves a problem in algebra, works an original exercise in geometry, or demonstrates the proof of a theorem.

4. The habit of making drawings representing abstract situations that can be pictured by diagrams and graphs.

5. The habits of estimating in advance the approximate answer to a problem involving computations; of deciding whether the answer found is reasonable.

6. The habit of checking results. In practical work checking is not only desirable but necessary. Surveyors and engineers must check their work constantly. Sometimes the check involves as much labor as the work checked, but they consider it worth while. Checking is therefore a valuable habit to cultivate in school mathematics. This does not mean that every process has to be checked or that the pupil needs the check to be convinced that his work is right. However, often the check offers training as valuable as the work which is being checked. In fact, some checks may be justified merely on the ground that they offer such training.

In arithmetic, computations are usually checked by means of the inverse operations or by reversing the order of the operation. Thus, the sum of a column of figures is checked by adding the column in the opposite direction. A product is checked by interchanging the factors and multiplying. A subtraction is checked by adding the subtrahend to the remainder, and a quotient by multiplying the quotient by the divisor.

In algebra the fundamental processes are checked conveniently by substitution of values for the literal numbers in the given problem and in the result, or by performing the process a second time. Solutions of equations are checked by substituting the roots in the original equations.

Objections to checks are aimed mostly at the wrong uses of checking. For example, when practice is required in the application of a principle of algebra, it is unwise to stop after each exercise to check the correctness of the result. Such interruptions may seriously interfere with the process of assimilating the principle to be learned. Checks are also out of order when it is aimed to develop speed. The checking of fractional roots of equations may involve work too elaborate and time consuming, and it may not leave

enough time for the practice needed to accomplish the major purpose, which is to develop ability in solving equations.

In general, checks should be avoided when they are a useless burden to the pupil and when they interfere with the real aims of the work to be accomplished. They should be used when they are understood by the pupil and are helpful to him in doing his work. Surely any pupil who wants to know whether his answer is right should be able to check it by some convenient short method. He will gladly use checks that will save him the trouble of doing his work a second time and in the same way.

Teaching the habit of planning in preparation for home study.—Effective study depends much on the physical and psychological conditions under which it is done. In the classroom it is usually easy to have conditions such as to make studying effective; but in the modern home, unavoidable distractions and diversions are so frequent that it seems often almost unreasonable to expect pupils to concentrate and to keep up interest in study. It is therefore of greatest importance that they should be shown how to make their home environment favorable to study. This may be done by calling attention in class to conditions that are conducive to study, as well as to those that interfere with study. Among other things, the pupil must be taught in school that improper lighting, heating, and ventilation lead to early fatigue; that he must clear his desk or table of all materials which might distract attention or otherwise interfere with concentration; and that he must bring to the study the needed paper, books, and instruments. This calls for careful planning. Pupils frequently fail to do the required work because they run out of notebook paper or other materials. The habit of planning and of keeping on hand an ample supply of notebook paper and other needed materials is therefore an important one to cultivate.

Teaching pupils to begin work promptly.—Many pupils use more time getting ready to work than doing the work after they start. When the necessary materials are on hand and when working conditions have been made right, no time should be lost in starting. It is better to take more time for play and to begin work promptly than to stop play early and to waste time getting ready. "Play hard

while you play but concentrate and have the will to learn when your studying begins" is a valuable principle to emphasize in teaching good habits of study.

The habit of beginning promptly can be cultivated in the classroom. Thus, when the bell rings, pupils should start working without waiting for instructions from the teacher. When an assignment has been made, the teacher should first of all see that everybody begins readily. When an announcement is to be made, prompt attention should be demanded of all. When a pupil is asked to report, he should respond promptly. If pupils have not acquired this habit, their attention should be called to it. They should then be carefully observed and trained until they develop the habit.

Paying close attention.—This habit is particularly important when difficulties are explained, when instructions are given, when assignments are made, and when the preview of a new unit is presented. Even a moment of inattention might be fatal. The suggestions made by the teacher may seem perfectly clear to the pupil at the time they are given, but only those who pay close attention will be able to recall them later when they do the assigned work. The others often fail entirely to make the connection between the suggestions and the tasks to be accomplished. It is important that the teacher's assignment always be brief, clear, and to the point. After that, the responsibility should rest with the pupil. When pupils have learned to pay close attention, it should not be necessary for the teacher to repeat assignments or suggestions.

Many pupils do not know what "paying attention" really means. A teacher giving the preview of a unit to a class in geometry noticed a boy playing with his ruler. He paused and said, "John, you are not paying attention." John promptly replied, "I am paying attention." The teacher went on with the discussion, and John continued to play with his ruler but kept his eye steadily on the teacher. He interpreted the teacher's rebuke to mean that so long as his eye was fixed on the teacher he was paying attention, regardless of what else he may be doing. The teacher, in dropping the matter, missed an opportunity to show what he meant by "paying attention." The pupil should have been made to understand that attention is unsatisfactory if it is divided between the ruler

and the teacher's explanations, that attention of the correct type means undivided attention, and that nothing else would be accepted.

Good listening.—The habit of good listening is important for successful work in mathematics. Attention was called previously to the lack of this habit. In many of the school subjects the pupils spend much time writing down notes on discussions or lectures; and later, at home or during the study period, they memorize these notes. It is natural for them to employ the same method in mathematics if a different type of training is not given. The difficulty with the method when used in mathematics is that pupils spend their class time writing down proofs of theorems and solutions of problems, only to find later that they are unable to read their own notes understandingly. They then memorize the notes whether they understand them or not. Gradually these pupils develop the questionable habit of drifting, idling, and postponing all serious efforts to understand what is said in class to a later time when they hope to be able to go over their notes more carefully. Usually they do not find the time to do this careful studying.

For this reason, class discussions in mathematics should be carried on only as long as every pupil is attentive and participates. No notes should be taken on what is said, but ample opportunity should be given for questions to make sure that the work is understood. Occasionally the class should be asked to reproduce discussions either orally or in writing. The written paper will help the teacher determine to what extent individual members of the class are able to profit from listening to a discussion.

Developing a correct critical attitude.—Statements made and questions asked by individual pupils should always be carefully considered by all the members of the class. Unless they hold themselves personally responsible for the answer, a situation easily develops where only one pupil and the teacher are engaged in a discussion. The class idles, and time is wasted. A statement made by the poorest pupil should receive just as courteous consideration by everybody as one made by the brightest. Only too often the former is ridiculed while the latter's opinion is accepted without challenge. A critical attitude toward both must be insisted on if pupils are to learn to form and to depend upon their own opinions.

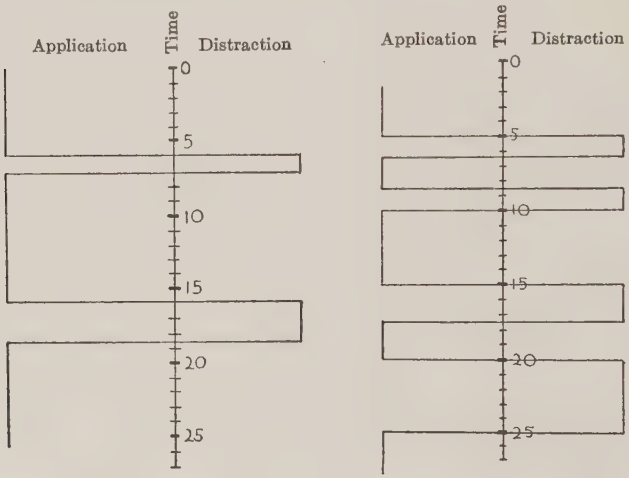
Training in sustained application and concentrated effort.—Some pupils apply themselves only to matters which appeal to their interest. When they are not being entertained, when a real difficulty is to be overcome, they soon cease to put forth real effort. They may appear to be studying, but, in fact, they are marking time. They are satisfied with the minimum performance, and lack the necessary mental energy and resourcefulness to continue farther. These pupils have to be taught to assume greater responsibility, to apply themselves more thoroughly, and to try every available device or suggestion in order to get the assigned work started and completed. "What is your method of attack?" "What have you tried?" "What does the problem say to you?" These and similar questions asked by the teacher are to be answered by the pupil who always says, "I can't do anything." If he knows that he is expected to make progress and that he may be called on at any time to report on what he has done, he soon learns to apply himself to his work.

The purpose of asking questions like those given above should be to teach pupils to ask themselves similar questions when a difficulty arises. Some pupils ask questions day after day without making the least effort to find their own answers. If they are allowed to keep this up, they soon become entirely dependent upon the teacher. If the teaching has been well done, if the assignment is carefully made, and if sufficient evidence has been obtained to show that the pupil is ready for study, there should be but little occasion for questions. On the other hand, after each explanation by teacher or pupil, ample time should be allowed for questions until every pupil feels that he is sufficiently prepared to proceed. Some pupils do not ask questions at all. It is important that they be encouraged to ask questions about things which they do not understand, but it is equally important that they should give the matter some thought before they speak.

A very good device of enabling the teachers to secure objective evidence regarding pupils' habits of giving sustained attention is given by Morrison.[1] He recommends the "application profile" as a means of recording how the individual applies himself to study, and shows how it may be used as a basis for training in sustained application. The profile is a graphical record of the extent to which

[1] H. C. Morrison, *The Practice of Teaching in the Secondary School*, chap. ix.

the pupil either applies himself or is distracted. The shifting from one phase to another is noted, and the time of each phase is recorded. The diagrams below illustrate how the graph, or profile, appears when finished. The one on the left shows the record of a pupil who applies himself thoroughly. He needs no further training. The case of the pupil recorded in the other diagram is very different. He is "fidgety" and is in great need of training. His case needs to be studied carefully by the teacher to determine the cause and the necessary remedial steps to be taken.



The value of the profile is greatly increased if notes are taken on the pupil's behavior during the time of observation. The record then shows what the various distractions are. The profile can be used in individual conference with the pupil; or it may be discussed with the class as a whole, in which case the names of individuals whose profiles are being presented are not to be made known. The important thing is to bring to the attention of all pupils the value of good habits of sustained application and to show how the habit may be developed.

Good reading habits.—Attention has been called previously to the fact that, in general, high-school pupils do not possess good reading habits. They read words but fail to get the meanings or the thought. They read a whole paragraph or page without being able

to recall what they have read. When progress in a course depends on ability to read, these pupils soon fall behind, as the following illustration shows: A teacher reported to the supervisor a boy as a hopeless failure in geometry. The supervisor summoned the pupil for a conference, at which the following conversation took place:

Supervisor: Your teacher tells me that you have difficulty in mathematics. What seems to be the trouble?

Pupil: I don't know. I don't understand any of it.

Supervisor: You mean to say that you understand absolutely nothing in the course?

Pupil: Yes. I can't make any sense of it.

Supervisor: Are you studying? Are you trying as you should?

Pupil: Yes. I read as much as the others in the class. I work harder than most of them, but I guess I don't understand it so well as they do.

A book was secured, and the pupil was asked to read aloud the lesson for the next day. He read a theorem fluently and rapidly and was asked to stop.

Supervisor: Do you understand that?

Pupil: No. I don't know what it means.

The pupil was asked to read again from the beginning. He started to read as rapidly as before but was stopped after the third word.

Supervisor: Do you understand that?

Pupil: Yes. I do.

Supervisor: Read on, please.

After reading several words the pupil was again stopped and was asked, "Do you understand that?" Again he said that he understood what he had read. This method of reading and questioning was kept up until the end of the statement was reached. The pupil then said that he understood all of it. He continued to read the proof in the same manner, one statement at a time, always reflecting after each. When the end of the proof was reached, it was found that he understood all but two minor details which he was told to assume as true for the time being.

This little lesson in reading was sufficient to put the pupil on the road to success, although, according to his own admission and his teacher's statement, he had been hopelessly lost in the course. The difficulty was not in the subject, in which no help was needed

or given, but arose from the lack of good reading habits. Having had this brought to his attention, the pupil saw the handicap under which he was working. So long as reading meant simply running rapidly over the content of the page, once or several times, progress had been impossible. As soon as he learned to read properly, his work improved. The pupil who has developed right reading habits will eventually become independent of the teacher.

Reading high-school mathematics involves a number of activities, any one of which, if not performed, may be fatal to success. Each calls for special teaching and training. In the case described in the foregoing paragraphs the cause of the difficulty was failure to pause, to reflect, to reread a passage when it was not clear, or to summarize consecutive passages. The pupil's reading habits were poorly developed. He read everything at the same rate, the simple material as well as the difficult. He thought that he was doing exactly what the other pupils were doing who were reading with understanding. Nobody had ever helped him to analyze his own peculiar difficulty because his teachers took correct reading habits for granted. Finally, the pupil came to the conclusion that the subject was too difficult for him and that it was useless to try further.

The following are reading habits which are essential to success in mathematics and in which training is required:

1. Reading the paragraph as a whole without trying to understand difficult words or sentences.

2. Reading again with the topic, problem, or central theme in mind. Every sentence read must now contribute something to the clearness of the topic. The pupil must constantly be aware of the contributions which the sentences or paragraphs make to a topic.

3. Pausing frequently to reread, to summarize, to generalize, to supplement, to apply, to reflect.

4. Making certain that the meaning of all words is clearly understood, and that new concepts are clear. This may call for the use of index and dictionary.

5. Varying the rate of reading according to the value, importance, or difficulty of what is read.

6. Comparing the data in a verbal statement with the corresponding facts illustrated in a drawing.

7. In reading constructions in geometry, or in symbolic notation, to perform each step as it is read.

8. Paying careful attention to footnotes, hints, and suggestions.

9. Associating and supplementing problem situations with knowledge acquired through former experiences.

10. Making sure in reading symbolic notation that the symbols are recognized and clearly understood.

The foregoing habits show that reading in mathematics must be intensive. Carelessness of omitting even one word often changes the meaning of the whole sentence.

Directed study offers excellent opportunities for training in correct reading. When the teacher observes the pupils as they read silently, he will frequently discover reading situations with which the majority of the pupils have difficulty. The following remedial method has been very effective: All class work is stopped and silent reading gives place to oral reading. A pupil is asked to read the sentence aloud. The teacher then shows how to make an analysis of the passage to locate the difficulty and suggests ways and methods of eliminating it. He also calls attention to special habits that particular pupils should learn to use to understand the sentence. The next step is to assign a paragraph, or a section, of the textbook for silent reading, with instructions to pupils (1) to analyze their own difficulties as they arise; (2) to be prepared to ask definite questions on matters that may not seem clear; and (3) to organize the content in such a way as to be able to give a clear oral report on what they have read. The oral report is called for as soon as all questions are answered. This step is effective because it places responsibility for results upon every pupil and makes reading intensive.

The discussion in the foregoing paragraphs has shown how oral reading may be used to develop correct habits of reading. However, oral reading should not be overdone. It aims to give instruction in reading habits and has served its purpose when it has shown how pupils may learn to do effective silent reading. The latter is the method used in nearly all work and study.

Clean-cut and neat writing.—Much of the studying in mathematics calls for written work. From the standpoint of mathematics it is important that the written work should always be clean cut and neat. Neatness in form is of great assistance to correct thinking. Carelessness in writing is only too often the cause of errors.

It may even result in failure. The habit of poor writing may lead to habits of poor mathematical thinking. It is essential, therefore, that pupils learn from the beginning to do neat work. An easy cure for careless writing is obtained by letting the pupil complete the assignment of a piece of written work, and then insisting that it be re-written in desirable form. Not many such experiences are needed to induce the pupil to do better. No poorly written paper should be accepted even though the work be done correctly so far as it concerns the mathematics involved. It is a good plan to exhibit on the bulletin board especially well-written work to stimulate the pupils to improve their writing. The use of scratch paper should be abolished, because it is probably one to the greatest causes contributing to carelessness. The following directions have been found helpful in improving written work:

a) Plain No. 6 notebook paper is to be used. Ruled paper is objectionable because the lines on the paper interfere with the lines in geometrical drawings.

b) On each paper the pupil's name and the date should be written in place assigned for that purpose, e.g., in the upper right-hand corner.

c) The year or grade of the class—for example, "Math. I"—and the page or pages of the book should be written in the upper left-hand corner.

d) Parallel to the left edge of the page and about one inch from it, a line should be drawn to leave a margin for marks and criticisms by the teacher. The pupils should write on this margin only the page and problem number.

e) The perforations should always be on the left side of the paper, and pages should not be torn from the notebook.

f) Only one side of the paper should be written on.

g) All work should be written with a sharp pencil (medium hard, never soft). Few pupils are able to make good drawings in ink.

h) Work should not be crowded on the page.

i) All drawings entered into the notebook should be made with ruler and compasses.

j) All answers should be marked plainly.

For work in algebra the page should be divided into two parts by a vertical line, as shown below. All algebraic work is then written on the left side of the line of division. All arithmetic computations that have to be written should be to the right in the same horizontal position.

All written computations should be worked on regular notebook paper in the same way and with the same care as all other written work. The teacher can often get a very good understanding of the pupil's difficulty by examining his computations in detail. Knowing this, the pupil will try to solve his problem correctly at the first

SAMPLE BLANK FOR WORK IN ALGEBRA

Name: John Anderson Date: May 3, 1928

		Computations
Page 60 No. 3	$r=6$ $A=\pi r^2$ $=\pi\times 36$ $=113$ approximately $\therefore$ the area is 113 square inches.	3.14 36 1884 942 11304
No. 4	$A=84$ $A=\pi r^2$ $84=3.14r^2$ $r^2=\frac{84}{3.14}$ $=26.7$ approximately $r=5.1$ approximately	26.7 3.14)84.00 628 2120 1884 2360 2198 162 5.1 $\sqrt{26.7}$ 25 101 170 101

attempt—a habit which will make him careful and exact in his work. Thus, neatness in writing will contribute to accuracy in computation.

A helpful device for teaching pupils to arrange geometric proofs and constructions in a clear and simple manner is the "theorem paper" (p. 108). The theorem is written horizontally across the top of the page. Below this a two-inch strip is marked off. This is divided into two parts by a vertical line. On the left side the pupil enters the facts given and to be proved. On the right the diagram is made. The remaining part of the page is bisected vertically, the left side to be used for the statements and the right side for the reasons. The paper may be used very effectively for several days at the beginning of the study of plane geometry. However, when every pupil in the class understands clearly the form in which the

proofs are to be written, plain notebook paper should be used, the pupil dividing the page as shown on the theorem paper.

THEOREM PAPER

Name:_____ Date:_____

Theorem:_____

Given:_____ DRAWING

To prove:_____

Proof:

STATEMENTS	AUTHORITIES

Effective reviewing.—It is commonly assumed that pupils have correct habits of reviewing. Review lessons are frequently assigned on work studied in class, especially in preparation for a test. However, many pupils have no conception of what is expected of them

when a review assignment is made. Indeed, some leave the classroom with the feeling of "no lesson for tomorrow." Others understand the assignment to mean that they are merely to repeat the old work in the old way. To determine the extent of the pupils' knowledge of the correct method of reviewing for a test, a teacher announced to his class, "Tomorrow we shall have a test. We shall spend today's class time reviewing." He gave no further suggestions and watched the class to see what the pupils would do. Very few seemed to know how to review. Some turned the pages of their book rapidly, reading superficially, and not getting anything in particular from the reading. In a few minutes they had finished and were idle. Others were reading one page over and over again. Still others, having met some difficulty, spent the entire period in the effort of overcoming it. It was apparent that they had no appreciation of the value of time, for they had but fifty minutes in which to review the work of three weeks. Several pupils had to be cautioned because they were not studying at all. At the end of the fifty minutes two pupils said that they had finished and thirteen said that they had reviewed more than half of the unit. Apparently the class did not review successfully under most favorable conditions. They failed to cover the ground in fifty minutes, which is more time than is usually allowed for a review for an announced test.

When the review is postponed to the day preceding a test, it cannot be satisfactory. The pupil then resorts to cramming. It is no wonder that he retains very little after the test has been taken.

What is needed is definite instruction in methods of reviewing. Otherwise there will be tremendous waste of the pupils' time in school and out of school. A review should always give a new and better view. Every day the pupil must go over important parts of the unit. If this is done, he will be well prepared to take the final test, when he reaches the end of the unit, without any special review assignment.

This type of review is very different from the meaningless review assignment at the end of a chapter, which is practically a preparation for a test by cramming material which should have been mastered from day to day. Reviews should never be confused with drills or cramming. A systematic review consists of a topical and

logical organization of whatever part of the unit has been studied. It starts when the unit is begun, and continues daily until the unit is finished. The pupil reaches the end of the unit with a clear, complete, logical organization in mind. Occasionally a class period or a part of a period should be set aside to give instruction in ways of reviewing and to determine whether the pupils know how to review.

The following example gives suggestions to the pupil to show him how he might review work in plane geometry. The procedure can easily be employed in other work.

Each time you learn a new theorem write the statement into your notebook. (When the textbook gives a list of theorems at the end of each chapter this copying is not necessary.) In your daily review turn to this list. Beginning with the first theorem read just far enough to recognize the theorem and finish the statement mentally.

Make a mental picture of the diagram with all of the auxiliary lines. If you do not feel sure of yourself turn to the textbook diagram to note omissions or mistakes.

Recall the method of proof.

Recall the important steps in the proof. If a proof is really understood it will be easy for you to recall the details whenever a complete proof is required.

Reviewing daily by this method keeps your attention on the whole unit which you are studying and goes far toward making it your permanent possession.

The responsibility for reviewing rests with the pupil, just as responsibility for teaching rests with the teacher. The review should constitute part of the daily home assignment. In fact, it should be the most important part. The teacher must keep in mind that the pupil is busy with many tasks and does first that which he knows will be called for first. Hence, it will be necessary to have some kind of check to make sure that the daily review is not neglected. Every pupil should be prepared at any time to give a brief, well-organized oral report on what he has learned in the particular unit that is studied. Such reports may be called for frequently at the beginning of the class period. They should usually take only a few minutes. If the pupil reviews systematically, very little of his time will be required each day to bring his organization up to date. Or, the teacher may ask in succession such definite questions as the

following: What are the methods of proof taught in this unit? What methods have been used to solve simultaneous equations, or quadratic equations? What kind of polynomials are you able to factor? What are the important theorems or algebraic principles that you have learned in this unit? What are the new terms that have been used? Questions like these may also be asked frequently in connection with new work. For example, if a new theorem is studied in which a former theorem is used, the teacher may ask: "What method was used in proving this theorem?"

Developing the habit of organizing instructional materials.—It cannot be assumed that pupils know how to organize without definite instruction and practice in organizing. Furthermore, organizing in one subject, as in English, is not necessarily the same as organizing in another subject. Pupils must be shown what a logical organization in mathematics is like. It should conform to common usage; i.e., it should first state the title of the unit. The major topics should be marked "I," "II," "III," etc., the first subtopics "A," "B," "C," etc., the second subtopics "1," "2," "3," etc.

The following are some of the major topics under which mathematical materials may be organized:

I. Usefulness of the unit
II. Concepts
III. Principles taught
IV. Processes learned
V. Constructions and geometric skills
VI. New methods of proof or solution
VII. Applications
VIII. Algebraic methods and skills
IX. Arithmetical skills

Each of these major topics will generally be subdivided. For example, in the unit on quadrilaterals one group of principles states the properties of parallelograms, and another the conditions under which quadrilaterals are parallelograms. It is evident that the outline of the various topics does not necessarily follow the arrangement of the book, which might stress the psychological more than the logical. The difference between the two types of organization is illustrated in the following example.

In the pedagogical organization of the unit on "Rational Inte-

gral Functions," new terms are introduced throughout wherever they are needed; but in the logical organization they are all collected under one topic, such as "New Concepts." In the same unit are taught various methods of finding the value of a given function and of solving certain types of equations. In the logical organization they are all collected under one topic, such as "New Methods." Furthermore, various types of functions are distributed throughout the unit. They may be grouped together logically as "Functions." In a number of places in the unit graphs are used to illustrate methods of solving problems. They are arranged topically under the heading "Uses of Graphical Methods."

In a complete logical organization every phase of the unit must be accounted for. In the daily organizing of the materials by the pupil, he should scrutinize carefully the textbook to make sure that he has provided a place for everything in the unit that is essential.

Incomplete and incorrect written organizations, after being criticized by the instructor, should be returned to the pupil to be put into acceptable form. The organizing of units gives excellent training in written expression.

The following example[1] illustrates the two types of organization for the unit on angles. The pedagogical organization shows the paragraph headings as given in the textbook. The logical organization shows how the material may be organized in the form of a logical outline.

UNIT ON ANGLES

Pedagogical Organization	Logical Organization
I. How angles are used 1. Meaning of angles 2. Uses of angles 3. Size of angles 4. Symbols to denote angles 5. Classification of angles II. Measuring angles with a protractor 1. The protractor 2. Units of angular measurement 3. Sum of the angles of a triangle	I. Concepts: angle, vertex, side, size of angle, right angle, straight angle, acute angle, obtuse angle, degree, minute, second, perpendicular lines, parallel lines, adjacent angles, supplementary angles, complementary angles, opposite angles, alternate interior angles, interior angles on the same side, isosceles triangle, equilateral triangle, right triangle II. Symbols: $\angle A$, a°, a', a'', $AB \parallel CD$, $AB \perp CD$

[1] E. R. Breslich, *Seventh-Year Mathematics*, chap. iii.

UNIT ON ANGLES—*Continued*

Pedagogical Organization	Logical Organization
4. Measurement of angles in surveying 5. Measurement of angles in navigation 6. Adjacent angles 7. Perpendicular lines 8. Supplementary angles 9. Complementary angles 10. Opposite angles III. Drawing angles with a protractor 1. Drawing an angle of given size 2. Drawing an angle equal to a given angle 3. Angles in isosceles and equilateral triangles IV. Parallel lines 1. Measuring of parallel lines 2. Symbols for parallelism 3. Drawing parallel lines 4. Angle pairs formed by three lines	III. Principles: A. Angle pairs 1. The sum of two supplementary angles is 180° 2. The sum of two complementary angles is 90° 3. Opposite angles are equal 4. If two parallel lines are cut by a transversal, the corresponding angles are equal and the alternate interior angles are equal B. Angles of a triangle 1. The sum of the angles of a triangle is a straight angle or 180° 2. If two angles of a triangle are equal, the triangle is isosceles 3. The acute angles of a right triangle are complementary IV. Drawing and Measuring with instruments A. Protractor in— 1. Measuring angles 2. Drawing angles 3. Drawing parallel lines 4. Making designs B. T-square, parallel ruler, and others V. Uses of angles in— A. Surveying B. Astronomy C. Navigation VI. Arithmetic A. All fundamental processes B. Changing degrees to minutes and seconds C. Adding and subtracting arithmetical values of angles VII. Algebra A. Solving equations in one unknown as $x+2x+5x=180$; $x+20=90$, $x+50=180$ B. Solving verbal problems relating to angles C. Studying relationships, as $a+b+c=180$, $a+b=90$ VIII. History of mathematics: The division of angles into degrees, minutes, and seconds

Summary.—The chapter has shown the importance and need of teaching pupils how to study and how to improve their habits of study.

There are general study habits and strictly mathematical study habits. Among the mathematical study habits the following are the more essential:

1. Using symbolic statements in place of verbal statements
2. Ascertaining the meanings of mathematical terms which are used in reports and discussions
3. Keeping in mind the known and unknown facts of a problem
4. Using diagrams and graphs to visualize abstract relations
5. Pre-estimating the answer to a problem
6. Checking the results of problems

The following general study habits have been discussed:

1. Planning for home study
2. Beginning work promptly
3. Paying close attention
4. Good listening
5. Assuming a critical attitude
6. Application and concentration
7. Good reading
8. Neat writing
9. Effective reviewing
10. Organizing the materials of the unit

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CHAPTER V

THE MATHEMATICAL EQUIPMENT: ITS USE AND CARE

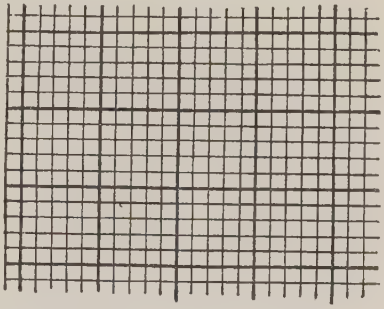
The need of equipment.—There is a growing feeling among the better teachers of secondary-school mathematics that the essential work of all courses should be done in the classroom if every pupil is to be reached. There is a tendency toward the laboratory method and supervised, or directed, study, to have the pupil do his work under the direction of the teacher, and to let him learn by studying and by doing rather than by imitating and listening.

To accomplish this a certain equipment is an absolute necessity. The importance of this has not always been emphasized or even recognized. Our modern secondary schools are housed in magnificent buildings. The shops are equipped with the most improved machinery. The laboratories contain the very best apparatus, but in the mathematics classrooms one looks in vain for anything like a working equipment. A few pieces of crayon, two or three erasers, some twine, and a few yardsticks furnished free of charge by some enterprising dealer in hardware and paints, make up the total equipment with which teachers and pupils are expected to do their work.

Responsibility for this deplorable condition sometimes rests with the administration that fails to recognize the necessity for a complete modern equipment in teaching mathematics, and sometimes with the teachers or supervisors who either have failed to impress the administration with the needs of the department or are ignorant of the amount and kind of equipment that should be provided. The matter is important enough to deserve the most careful consideration of every teacher, supervisor, and administrator.

For convenience, the mathematical equipment to be described in the following pages has been classified in three ways: the pupil's individual equipment, the classroom equipment, and the mathematical library.

The pupil's notebook.—Nearly every department of the school requires written work. It is important for more than one reason that all written work be preserved in a notebook. It gives pupil and teacher an exact record of what has been accomplished. Occasional inspection of the notebook shows how well each pupil performs his daily tasks, and thereby contributes much to accuracy and neatness. It is useful to the pupil in reviewing daily the essentials of the unit. It provides for individual differences, because when the rapid worker finds that he has time to spare he may keep himself profitably employed by completing unfinished assignments, or by checking results. In a similar way may the slow pupil spend some time during the study period or at home to make up deficiencies. Finally, when the unit is finished, the notebook gives the teacher a clear idea as to the character of the pupil's work of the entire unit.



Metric Squared Paper

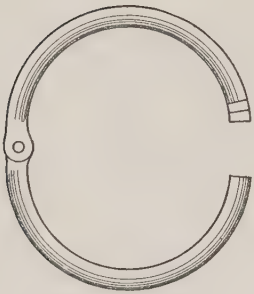
When several departments of the school require notebooks, they should agree as to uniformity of size and quality of paper. The notebook should then be divided into sections, each of which is to be reserved entirely for the work of a particular course. It has been found that the so-called "No. 6" size notebook, the pages of which are 8 by $10\frac{1}{2}$ inches, answers very well the purposes of all departments.

The mathematics section should contain "unruled" paper and "squared" paper. Ruled paper should not be used because the lines interfere with the clearness of geometric drawings and with the arrangement of algebraic or arithmetical computations. "Metric" squared paper is an excellent device for familiarizing the pupil with the decimal notation and metric system. It is very useful as a measuring instrument well adapted to all graphical work.

Notebook covers and papers should be fastened together with large rings about $1\frac{1}{2}$ inches in diameter, because, when small rings are used, it is difficult to turn the pages without tearing the perforations of the paper. Torn holes should be mended by pasting over

them gummed reinforcement rings, a supply of which should be kept by every pupil.

It is advisable that the younger pupils, especially those of the seventh grade of the junior high school, leave their notebooks in the classroom. At the end of each class period monitors assigned to the various rows of pupils should collect the books and other equipment, and place them where they will be ready for use the following day. During the intermission preceding the next class period the books should be distributed by the same monitors. Every pupil



Notebook Ring

Gummed Reinforcement Rings

will then have the materials he needs and will be ready to start his work promptly as soon as the class period begins. If a pupil has to rewrite some of his papers, or wishes to complete at home or in the study room a task begun in the classroom, he may, with the consent of the teacher, take that particular part of his work with him and put it back into the notebook the following day.

At convenient intervals, usually at the end of a chapter, or unit, the written work should be fastened together with clips and left with the teacher for inspection. Work that is incomplete, incorrect, or slovenly done should then be returned to the pupil for completion or rewriting.

The ruler.—The ruler is used as a straightedge for drawing straight lines and as a measuring instrument to determine the

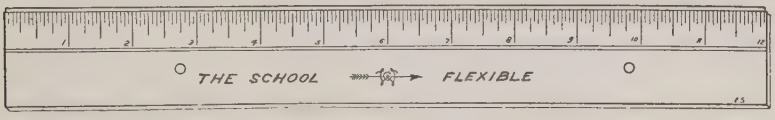
lengths of lines. It should have a brass edge on one side, and it should be perforated in two places so as to fit exactly over the notebook rings. By passing the rings through the perforations the pupil may attach the ruler securely to the notebook, and the possibility of losing it is thereby greatly reduced. The ruler will be in almost constant use, since no free-hand sketches should be accepted in notebook work.

Paper Fastener

The protractor.—The protractor is used to draw and measure angles and to check the accuracy of geometric constructions. Protractors are made of cardboard, celluloid, or metal. Since the instrument is used often, it is important to choose one that will not wear out from use and with which accurate work can be done. Metal protractors are best, and cardboard protractors are least desirable. A metal protractor should be selected on which the divisions are marked with black lines, because black lines are easily distinguished, even when the instrument has become marked with finger prints.

To avoid losing the protractor, the pupil should form the habit of attaching it to the rings of his notebook when he is not using it.

The combination triangle.—In place of ruler and protractor a celluloid right triangle is often used. This is not as satisfactory for drawing and measuring as the two separate instruments, but it has

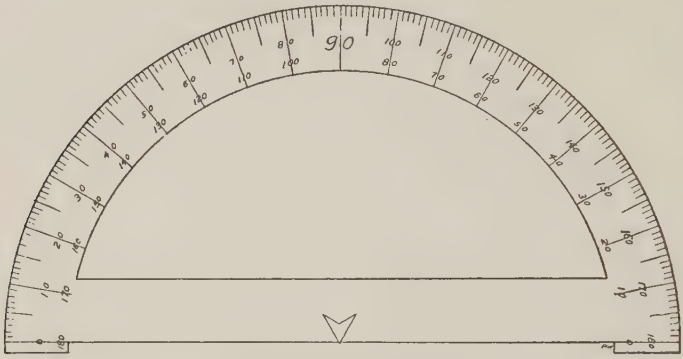


The Pupil's Ruler

the advantage of being convenient for drawing right angles and perpendicular lines.

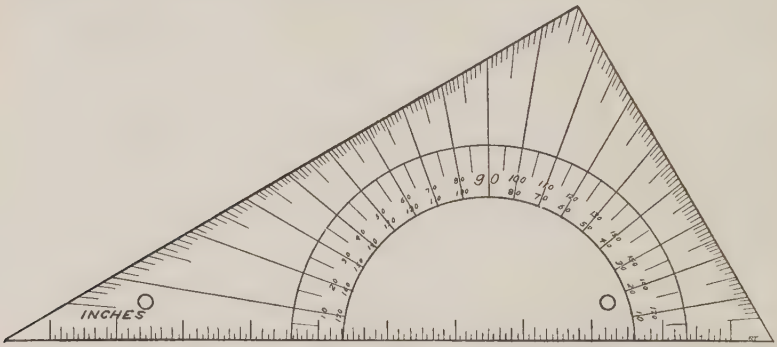
The compasses.—The compasses illustrated in the diagram (p. 121) give very good results. Moreover, the instrument is less expensive and at the same time more durable than some of the more expensive makes. It is easily carried by attaching it to the notebook rings with the other instruments. However, it is a

delicate instrument, and pupils should be shown how to use it correctly. They need to be taught how to slip the compasses over the pencil, to fasten or loosen the screw, to open and close the instrument, and to hold it in drawing arcs and circles.



THE PUPIL'S PROTRACTOR

The University of Chicago Press sells a complete set of ruler, protractor, and compasses, at a very reasonable price. They are placed in an envelope especially designed for the instruments, which can be pasted on the inside of textbook or notebook covers.



COMBINATION RULER AND PROTRACTOR

The textbook.—Every pupil should have his own textbook. Teaching him how to use and read the textbook is one of the important contributions which all subjects should make to the pupils' education.

The slide rule.—The slide rule is coming more and more into use as an instrument of computing and checking results. Gradually the business world is making it one of its most important computing instruments. Problems in trigonometry and physics can be worked out with the slide rule with a great saving of time, and the results of trigonometric and algebraic solutions can be quickly and easily checked with the rule. A student rule can be purchased at a very reasonable price. Each pupil should have his own instrument if he expects to do effective work.

Large demonstration slide rules can be purchased from the makers of rules. They are very valuable for teaching groups of pupils how to use the rule. Detailed suggestions as to using or teaching others to use the rule are given in a little textbook published by the University of Chicago Press.[1]

THE PUPIL'S COMPASSES

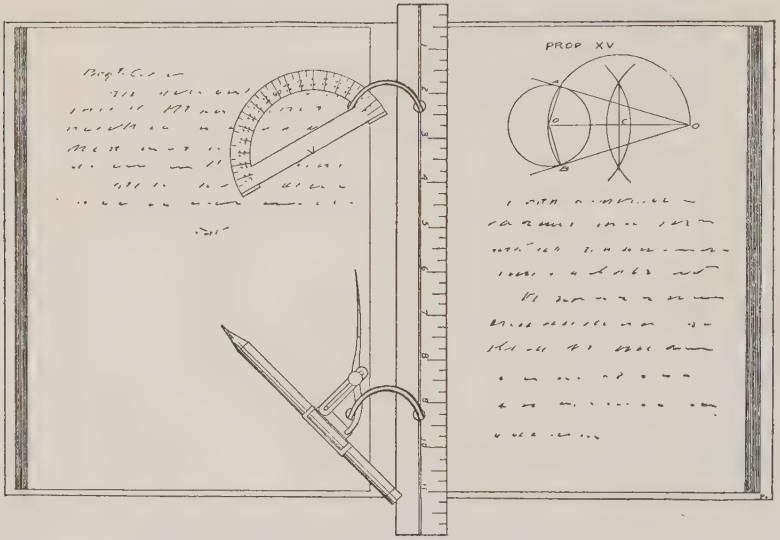
The materials and instruments described in the foregoing pages constitute a complete student equipment. It is important that every individual pupil be provided with a set of his own. Pupils should not be allowed to lend, borrow, or exchange materials during the class period. The practice always interferes with effective work. It disturbs pupils at work and often leads to inattention or even disorder. Pupils should be taught to cultivate the habit of giving a little thought to the required equipment before they come to the classroom.

Desks.—The pupil's desk should have enough surface space to allow him to work comfortably and to accommodate the textbook and open notebook. The aisles between the rows of desks should be wide enough to permit pupils to pass quietly and noiselessly to and from the blackboard, and the teacher to go from desk to desk to examine written work or to give individual attention. Failure to

[1] E. R. Breslich and Charles A. Stone, *The Slide Rule* (Chicago: University of Chicago Press, 1929).

allow sufficient space makes it impossible for the teacher to direct or supervise the work of the pupils.

Blackboards.—If conditions are to be ideal, all available wall space of a mathematics classroom must be used for blackboard space, because frequently the whole class should be working at the board. In every classroom there ought to be enough room to send at least half of the class to the board. While one-half of the class



Complete Pupil Equipment for Mathematics

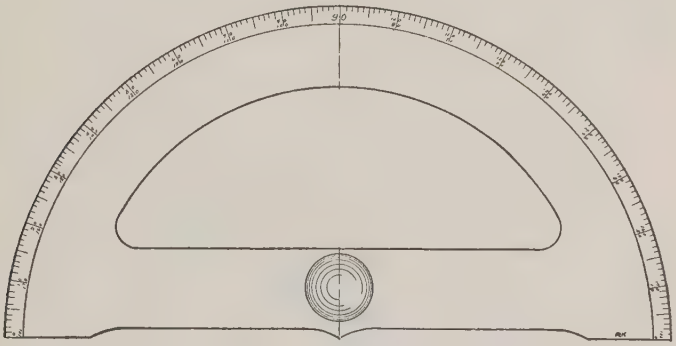
works at the board, the other pupils can do the assigned work at their seats.

Squared board.—One section of the blackboard should be cross-ruled for graphical work. It is essential that the squared board be placed where it is clearly visible from every part of the room. At the same time it should not be located where teacher and pupils do most of the written work. The sides of the large squares should be about 7 inches long. The color of the lines should not be white, because the white lines interfere with the lines of drawings and graphs. A dark brown color has been found very satisfactory.

A stationary cross-ruled blackboard is far superior to a black-

board "chart." It is almost impossible to do accurate work on a chart.

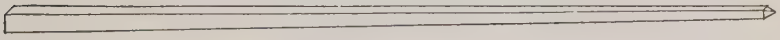
Blackboard protractor.—The protractor is used to measure and draw angles on the blackboard and to check geometric constructions. It is also needed to demonstrate to pupils the meaning and use of their own protractors. For this reason blackboard and pupil



Blackboard Protractor

protractors should resemble each other as much as possible. A classroom should be equipped with two or three blackboard protractors. They should always be kept in places where they are most convenient for use.

Blackboard Ruler



Blackboard Pointer

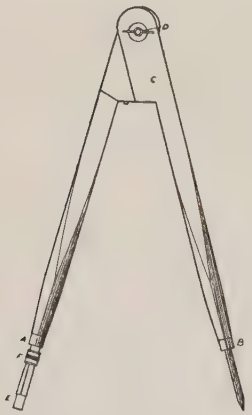
Rulers, pointers, and erasers.—The ruler illustrated above was designed and made by a pupil in the school shop. Its actual length is 30 inches, and its width is 3 inches. By means of the handle the ruler can be pressed firmly against the blackboard, which

prevents sliding. At least three of these rulers should be in a classroom.

The pointer was also designed and made by a member of a shop class. It is about $3\frac{1}{2}$ feet long. It is made out of hardwood and is very durable. Three pointers are sufficient for a classroom.

The classroom should be amply provided with erasers. Not more than two pupils should ever have to use the same eraser.

Blackboard compasses.—One of the best blackboard compasses is illustrated in the adjoining figure. For efficient work a classroom should be equipped with about a dozen compasses. Two or three of them should be placed where they are easily available for use. The others, when not in use, should be kept in the teacher's desk or in some other convenient place. The task of keeping compasses and other equipment in working order may be assigned to a pupil.



Blackboard Compasses

The need of blackboard compasses should be emphasized. One of the major aims of construction work on geometry is to teach the pupil to make a *construction* as accurately as he is able to make it. Hence the best instruments must be used. Whether it is done on paper or on the blackboard, a construction in geometry should satisfy all the requirements of an exacting mechanical-drawing course. This eliminates the use of string and crayon in place of compasses.

On the other hand, sometimes, as in the case of attacking original exercises, quick free-hand drawings are required. Pupils must, therefore, be taught to make good free-hand drawings without the use of any instrument. In both cases there is no place for string and crayon.

Filing-case.—Most modern classrooms have a built-in bookcase. If not, a steel case may be used. It is needed for storing reference books, materials, models, test papers, and notebooks.

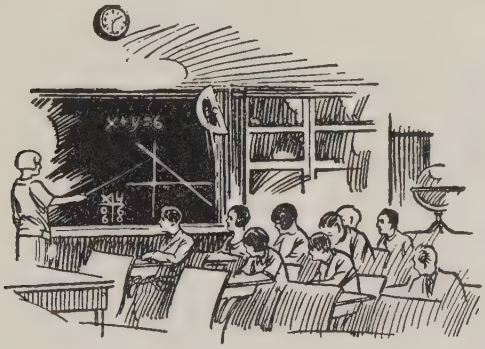
Pencil-sharpener.—If neat and accurate work is expected, no classroom should be without a pencil-sharpener.

The materials described on the foregoing pages form the "minimum" equipment required for effective classroom work in mathematics. The following are also desirable:

1. Models of the common solids needed to illustrate relationships between points, lines, and planes, in space.

2. Pieces of cardboard, sticks, and string, to be used in building up models illustrating geometric principles.

3. Several wooden triangles with acute angles equal to 45°, 30°, and 60°.



A Corner of a Modern Classroom, Showing Pointer, Protractor, Filing-Case, and Spherical Blackboard

4. A box of colored crayon.

5. Several pairs of scissors.

6. A spherical blackboard for the study of solid geometry.

7. A collection of pictures to illustrate the uses of mathematical principles in art, architecture, and engineering. Usually the pupils collect this material. The pictures should be mounted on uniform cards.

8. A large demonstration slide rule for teaching the use of the slide rule.

9. A bulletin board for posting assignments and exhibiting good work.

10. Surveying instruments, such as surveyors chain, pins, steel tape, poles, and transit.

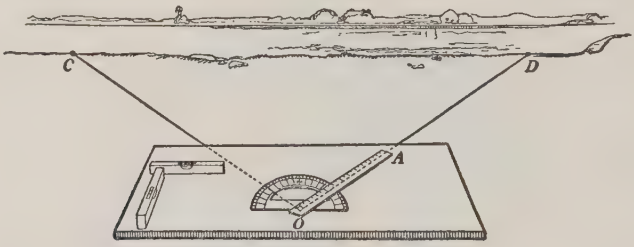
Homemade instruments.—Some good work in mathematics can be done with homemade instruments and models. The use of the models made by pupils usually arouses interest in making other and better models. Some pupils will make wooden and metal models in the school shops. Others will make them at home, using cardboard

or string. By keeping the best samples, the teacher will soon accumulate a very good collection of models.

The following directions show how to make an instrument for measuring angles in the horizontal plane:

"On a drawing board fasten a large protractor. By means of a pin stuck into the board at *O* attach a ruler so that it may move freely over the board when it is made to turn about *O*. A pin stuck in at *A* near the other end of the ruler may be used for sighting.

"The board may be placed on a tripod, or a table, and brought into horizontal position by means of two spirit levels attached as shown in the figure.



Homemade Instrument for Measuring Horizontal Angles

"To measure $\angle DOC$, sight in the direction of the side *OD* and take the reading of the protractor. Repeat this for the side *OC*. The difference between the two readings is the *measure* of the angle."[1]

The transit pictured on page 127 was made by a pupil of the University of Chicago High School.

An instrument for measuring angles in the vertical plane (p. 128) may be constructed as follows: Two boards are fastened at right angles to each other. The horizontal board is placed on a tripod and is kept in horizontal position by means of two levels. The protractor is fastened to the vertical board as shown in the drawing. A ruler is used for pointing the direction.

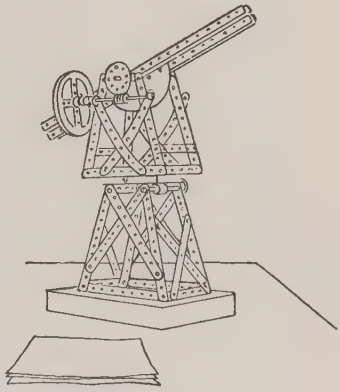
The use and care of the equipment.—With the mathematical equipment described in the foregoing pages it is possible to carry on satisfactory laboratory experiments. However, certain difficulties may arise in connection with the use of the instruments.

The first difficulty occurs when pupils lose, break, or forget

[1] E. R. Breslich, *Seventh-Year Mathematics*, p. 101.

some of their materials. They are then delayed in starting their work and are losing time. A small supply of rulers, compasses, protractors, and notebook paper may be kept by the teacher for the purpose of lending them to pupils in cases of emergency. Care should be taken not to let pupils form the habit of depending on the teacher to help them repeatedly when a little thought on their part could have avoided the necessity. Requests by pupils for materials should always be made before the class period.

Sometimes for days no use is made of some or all of the instruments. The thoughtless and easy-going pupils will soon come to class without them; and when the next experiment is to be performed, they will be unable to take part.



Homemade Transit

To avoid both difficulties, pupils must develop the habits of attaching the instruments to the rings of the notebooks when they stop working and of giving some attention and thought to the equipment before they come to class. Until right habits are established, occasional reminders by the teacher will be necessary.

Another difficulty arises when pupils do not know how to use the instruments. Definite instruction in the use and manipulation of all instruments is required. An effective method of teaching the use of an instrument is illustrated in the examples below. The procedure instructs the class as a whole but aims to teach the individual. One step is taken at a time, and all pupils respond immediately to each step. It is a case of learning directly by doing. Care must be taken to allow enough time between successive steps to ascertain, by glancing over the classroom, whether each pupil has made the correct response. All instructions must be brief and to the point.

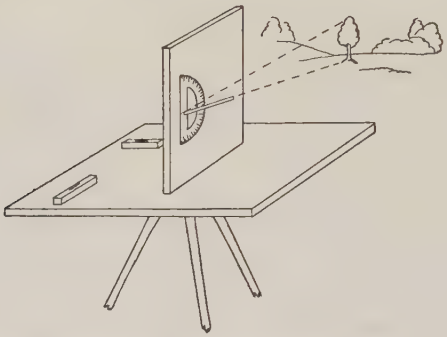
Example 1: To draw a straight line with a ruler.—Many pupils continue to have difficulty with this exceedingly simple problem because they have not been shown how to do it. For example, they are unable to hold the

ruler in a fixed position while they move the pencil point along the edge, especially when they draw straight lines on the blackboard. The following instructions are sufficient to show pupils how to draw straight lines:[1]

"Place your ruler on a sheet of paper and press down with your left hand like this. [As the teacher holds his left hand in position, he watches the class until every pupil has responded.]

"Move the pencil point along the edge like this. [Again the teacher watches each pupil.]

"Take away your ruler. You have drawn what is called a straight line."



HOMEMADE INSTRUMENT FOR MEASURING VERTICAL ANGLES

This is now to be followed by practice exercises consisting of drawing straight-line figures, such as triangles, rectangles, and other polygons. However, practice will be continued permanently if hereafter all straight-line figures drawn by pupils in their notebooks are made carefully with the ruler.

Example 2: To measure a line segment with a ruler.—The teacher sketches a segment on the blackboard and says: "Draw carefully a segment on notebook paper." He then marks the segment AB and says: "Mark it AB."

"Place your ruler under your line segment with the zero mark directly under A, like this." [The teacher is holding the ruler in this position and observes the class until everybody has responded.]

"Read the mark on the ruler which is directly under B."

"Write your result on AB like this." [The teacher writes his result near the midpoint of AB.]

Example 3: Measuring angles with the protractor.—The following instructions are given:

[1] *Ibid.*, p. 19.

The teacher makes a sketch of an angle on the blackboard and says: "Make a careful drawing of an angle like this. Be sure you make the sides long enough, about $2\frac{1}{2}$ inches long." It will now be necessary for the teacher to go about in the classroom to determine if the pupils' responses are correct. They should be told to make drawings similar to the sketch on the board because it will then be simpler for them to understand the statements made about them later.

The teacher continues: "Place your protractor on the angle so that the center falls exactly on the vertex, like this."

"Now turn the protractor until the zero mark falls exactly on one side of the angle, like this."

"Pass along the rim counting zero, ten, etc., to the other side of the angle and take the reading where the side intersects the rim."

Removing the protractor, the teacher says: "Write your result within the angle as shown in my drawing."

Example 4: Teaching how to use the compasses.—The teacher, standing before the class, shows how the compasses are slipped over the pencil. Then the pupils try it and practice until they can do it. Unless they are shown the right way, many will bend the compasses in the first attempt of trying. The teacher continues, saying: "When measuring with the compasses, tighten the screw but slightly until the required distance is obtained. Then tighten it firmly.

"Whenever you change to another length or when you close the compasses always loosen the screw first."

Instruction must also be given in drawing arcs and circles with the compasses. It takes some pupils a long time to learn to manipulate the instrument correctly. The teacher says: "Hold the compasses with one hand between your thumb and forefinger. Press down lightly on the sharp point. Now turn the compasses and keep on practicing until you can easily draw a smooth circle."

To some teachers, the foregoing exercises may seem too simple to deserve the detailed discussion that has been given. However, a few visits to classes in geometry will convince anyone that the problem is not simple. Many tenth-grade pupils are still unable to make satisfactory drawings either on paper or on the blackboard because they were never shown how to manipulate ruler and compasses when they used them in earlier work. Invariably time is wasted when the teacher has to go from pupil to pupil to instruct those who do not know how to work with the instruments. The

whole difficulty can be avoided by using the scheme of group instruction described above.

It is hardly necessary to add that similar instruction in the use of the blackboard instruments must be given before pupils are permitted to use them. Attention of the pupils should be called to the following: The blackboard compasses are easily broken either at the arms A and B (p. 124) or at the joint C if too much force is used. The screw at C should first be tightened slightly, just enough to open the arms to the place where the sharp points are as far apart as the required distance. The screw is then tightened sufficiently to hold the arms in place. It should be emphasized that the screw is now too tight to permit changing the distance between the sharp points. The force required to open or close the compasses when the screw is tight might break it, although the caps placed over the ends at A and B lessen the possibility considerably.

Before the crayon is inserted, the ring F should be pushed back. Forcing the crayon up too high bends and weakens the metal strips that hold it in position.

In drawing circles or arcs, the instrument should be held and manipulated with one hand. Considerable practice in drawing circles is needed.

The mathematical library.—Every classroom should have at least one bookshelf with different textbooks containing readings parallel to the topics discussed in class. For example, in geometry pupils are interested in different proofs of the same theorem. In algebra some books are clearer in presenting a certain topic or principle than others.

Moreover, to provide for the various interests which pupils develop there is need for books on mathematical recreations, for histories of mathematics, for copies of mathematical journals, and for mathematical tables. There is a demand for reading materials and supplementary exercises needed for those who wish to do more than the required work. In a certain sense, the extent to which pupils become really interested in the subject is a measure of the teacher's success. When pupils develop a genuine interest in the subject, they are anxious to go more deeply into certain phases of the subject than is possible with a classroom library. When they voluntarily undertake an independent piece of work, they are in need of ample

reading material. The school library is the best place for such material.

The mathematical library for high schools is still in its infant stage, not because there is a scarcity of suitable books and magazines but because that material has not been properly graded and classified to be readily adapted to pupils of different stages of mathematical education.

There must also be sufficient variety of reading matter to suit the inclinations and interests of the different pupils. Even those who do not care to undertake special projects should find much in the mathematical books that is of interest to them. Historical textbooks appeal to some. Others enjoy reading of the recreational type. Still others prefer the applications.

There is now available a considerable body of mathematical reading material of the recreational type. Some of this may be kept in the classroom, but most of it should be used to build a mathematical library. Pupils should be encouraged to spend their leisure time on this type of reading. It includes books on mathematical puzzles, stories, jokes, even poems, but also material of the more serious type, such as biographies of mathematicians and books on the history of the development of mathematics. The following list of books makes a suitable mathematical library:

HISTORIES OF MATHEMATICS

Allman, G. S. *Greek Geometry from Thales to Euclid.* London: Longmans, Green & Co., 1889.

Ball, W. W. R. *Short History of Mathematics.* New York: Macmillan Co., 1893.

Cajori, Florian. *History of Mathematics.* New York: Macmillan Co., 1905.

———. *William Oughtred.* Chicago: Open Court Publishing Co., 1916.

Fink, K. *A Brief History of Mathematics.* Chicago: Open Court Publishing Co., 1900.

Gow, J. *A Short History of Greek Mathematics.* Cambridge: Cambridge Press, 1884.

Karpinski, Louis. *The History of Arithmetic.* Chicago: Rand, McNally, 1925.

Miller, G. A. *Historical Introduction to Mathematical Literature.* New York: Macmillan Co., 1927.

Smith, D. E. *History of Mathematics* (2 vols.). Boston: Ginn & Co., 1923.

Smith, D. E., and Karpinski, L. *The Hindu Arabic Numerals.* Boston: Ginn & Co., 1911.

MATHEMATICAL RECREATIONS

Abbott, E. A. *Flatland.* London: Seeley & Co., 1884.

Ball, W. W. R. *Mathematical Recreations and Essays.* New York: Macmillan Co., 1925.

Clark, Mary L. *The Adventure of X.* Boston: D. C. Heath & Co., 1920.

Dudeney, H. E. *Amusements in Mathematics.* New York: T. Nelson & Sons, 1917.

———. *Canterbury Puzzles and Others.* New York: T. Nelson & Sons, 1919.

Evans, H. R. *The Old and New Magic.* Chicago: Open Court Publishing Co., 1906.

Guerber, H. A. *Myths of Greece and Rome.* New York: American Book Co., 1893.

Jones, S. I. *Mathematical Wrinkles.* Nashville, Tennessee: S. I. Jones, 1912.

Licks, H. E. *Recreations in Mathematics.* New York: D. Van Nostrand Co., 1917.

Schubert, Herman. *Mathematical Essays and Recreations.* Chicago: Open Court Publishing Co., 1898.

Slosson, E. E. *Easy Lessons on Einstein.* New York: Harcourt, Brace & Co., 1920.

Smith, D. E. *Number Stories of Long Ago.* Boston: Ginn & Co., 1919.

Weeks, Raymond. *The Boys' Own Arithmetic.* New York: E. P. Dutton & Co., 1924.

White, W. F. *Scrapbook of Elementary Mathematics.* Chicago: Open Court Publishing Co., 1910.

FAMOUS MATHEMATICAL PROBLEMS

Beeman, W. W., and Smith, D. E. *Famous Problems of Elementary Geometry.* Boston: Ginn & Co., 1897.

Klein, Felix. *Famous Problems in Elementary Geometry.* Boston: Ginn & Co., 1897.

Rupert, W. W. *Famous Geometrical Theorems and Problems.* Boston: D. C. Heath & Co., 1901.

Withers. *Euclid's Parallel Postulates.* Chicago: Open Court Publishing Co., 1905.

Young, J. W. A. *Monographs on Topics and Elementary Mathematics Relevant to the Elementary Field.* New York: Longmans, Green & Co., 1911.

FAMOUS MATHEMATICIANS

Frankland, F. W. B. *Story of Euclid.* London: G. Newness, 1902.

Smith, T. *Euclid, His Life and System.* New York: Charles Scribner's Sons, 1902.

BOOKS OF GENERAL INTEREST

Adams, Mary. *A Little Book on Map Projection.* London: George Phillip & Son, 1914.

Andrews, W. S. *Magic Squares and Cubes.* Chicago: Open Court Publishing Co., 1908.

Bragdon, Claude. *Primer of Higher Space.* New York: Alfred A. Knopf, 1923.

———. *Projective Ornament.* Rochester, N.Y.: Manas Press, 1915.

Breslich, E. R., and Stone, Charles A. *The Slide Rule.* Chicago: University of Chicago Press, 1928.

Brewster, G. W. *Common Sense of the Calculus.* Oxford, 1923.

Bruce, W. H. *Some Noteworthy Properties of the Triangle and Its Circles.* Boston: D. C. Heath & Co., 1902.

Carus, Paul. *Foundation of Mathematics.* Chicago: Open Court Publishing Co., 1908.

Conant, L. L. *The Number Concept.* New York: Macmillan Co., 1896.

———. *Original Exercises in Plane and Solid Geometry.* New York: American Book Co., 1905.

Fine, H. B. *The Number System of Algebra.* Boston: D. C. Heath & Co., 1901.

Hedrick, E. R. *Constructive Geometry.* New York: Macmillan Co., 1916.

Jaffe, Bernard. *Chemical Calculations.* Yonkers-on-Hudson, World Book Co., 1926.

Keyser, C. J. *The Human Worth of Rigorous Thinking.* New York: Columbia University Press. 1916.

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Morris, I. H. *Geometrical Drawing for Art Students.* New York: Longmans, Green & Co., 1926.

Morris, I. H., and Hustand, J. *Practical Plane and Solid Geometry.* New York: Longmans, Green & Co., 1916.

Public Education Commission of American Bankers Association. *Talks on Banking for Elementary and High Schools.* East Forty-second Street, New York.

Rich, F. M. *The Jolly Tinker.* New York: D. Appleton & Co., 1923.

Row, T. S. *Geometric Exercises in Paper Folding.* Chicago: Open Court Publishing Co., 1901.

Shireby, R. M. *Slide Rule Applied to Commercial Calculations.* New York: Isaac Pitman & Sons, 1922.

Spencer, William G. *Inventional Geometry.* New York: D. Appleton & Co., 1877.

Sykes, Mabel. *Problems for Geometry.* Chicago: Allyn & Bacon, 1912.

Thompson, Sylvanas. *Calculus Made Easy.* New York: Macmillan Co., 1911.

Vincent, H. D. *Vocational Arithmetic.* Boston: Houghton Mifflin Co., 1914.

Wentworth, George. *Machine Shop Mathematics.* Boston: Ginn & Co., 1922

Whitehead, A. N. *Introduction to Mathematics.* New York: Henry Holt & Co., 1911.

Wright, H. C. *Children's Stories of the Great Scientists.* New York: Charles Scribner's Sons, 1888.

SURVEYING

Baker, C. E. *Engineers Surveying Instruments.* New York: John Wiley & Sons., 1918.

Barton, S. M. *Plane Surveying.* Boston: D. C. Heath & Co., 1906.

Johnson, L. S., and Smith, J. B. *The Theory and Practice of Surveying.* New York: John Wiley & Sons., 1915.

Tracy, J. C. *Plane Surveying.* New York: John Wiley & Sons, 1925.

BOOKS AND JOURNALS FOR PUPILS AND TEACHERS

American Mathematical Monthly.

Mathematics in Secondary Education. Houghton Mifflin Co.

Mathematics Teacher.

Popular Science Monthly.

School Science and Mathematics.

Scientific American.

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Barnhill, John F. "The Slide Rule in Junior and Senior High Schools," *Mathematics Teacher,* XVII (October, 1924), 359–64.

Brown, Zena. "The Use of the Slide Rule in Mercantile Establishments," *School Science and Mathematics,* XXVII (April, 1927), 390–96.

Cajori, Florian. "A History of the Logarithmic Slide Rule and Allied Instruments," XXXII, *Science* (November, 1910), 666–68.

Dooley, W. H. "Practical Mathematics for High Schools," *Mathematics Teacher,* X (June, 1918), 181–84.

Fensholt, A. H. "A Slide Rule Constructed without Logarithms," *School Science and Mathematics,* XV (May, 1915), 417–21.

Gorsline, W. W. "The Slide Rule in Plane Geometry," *Mathematics Teacher,* XVII (November, 1924), 385–403.

Moore, E. H. "Cross Section Paper as a Mathematical Instrument," *School Science and Mathematics*, VI (June, 1906), 429–50.

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Shelly, S. L. "The Slide Rule in Business," *Mathematics Teacher*, XIV (May, 1921), 261–62.

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Stark, W. E. "Early Forms of a Few Common Instruments," *School Science and Mathematics*, IX (December, 1909), 871–74.

Stark, W. E. "Measuring Instruments of Long Ago," *ibid.*, X (January, 1910; February, 1910), 48–67, 126–39.

Stone, Charles A. "The Slide Rule in the Junior High School," *ibid.*, XXX (June, 1930), 645–50.

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CHAPTER VI

THE UNITARY ORGANIZATION OF INSTRUCTIONAL MATERIALS

Bases of organization.—The need of organizing subject matter for purposes of effective teaching or learning has always been recognized. It is evident that the character of the organization must be determined by the objectives to be achieved. For example, a body of mathematical facts and principles may be arranged in alphabetical order for reference purposes, or in order of difficulty to facilitate assimilation. It may be grouped topically for purposes of review, or it may be arranged in a strictly logical sequence to offer intensive training in logical reasoning.

The alphabetical arrangement has given us dictionaries and encyclopedias, and for reference purposes this type of organization cannot be surpassed. However, no teacher would consider the dictionary a suitable textbook for teaching spelling, or the encyclopedia as a textbook in mathematics. This would represent the subject in its least attractive aspect. It would violate all the laws of learning. It could not be adapted to the needs of the pupils. Thus, it is quite possible that a basis of organization which gives excellent results for one purpose might utterly fail when employed for other purposes.

The topical organization of subject matter.—As the name indicates, this plan selects a number of important topics of a subject and distributes the instructional materials among them. In studying the subject, the pupil concentrates on one topic at a time. Each may be mastered and tested separately without any thought to its relation to other topics and to the course as a whole. Thus, the first topic of a course in geometry might present the definitions of the subject. It will define such terms as "geometry," "form," "angle," "circle," "area," and numerous others. The aim is to dispose of all definitions before taking up the next topic. It is immaterial that some of the terms will not be used for months, that some will occur almost constantly and others very little, and that some are

simple and others complex. They are all treated together in one topic.

As another topic of geometry one might choose "rectilinear figures." It would deal thoroughly with the properties of such figures as straight lines, angles, triangles, quadrilaterals, and polygons in general; but it would avoid carefully all references to other topics, regardless of the fact that they could be made to contribute to a fuller understanding of rectilinear figures. For example, no proportions can be used, because proportions are discussed as another topic.

The topical form of organization has certain obvious disadvantages. When pupils are studying one topic intensely to the exclusion of the others, they frequently develop the deplorable mental attitude of getting through with a topic and dropping it. They lose the desire of making it their permanent possession. Indeed, having completed a topic and passed an examination on it, some pupils express resentment when they are called upon at a later time to reproduce subject matter studied and tested in previous topics.

Experience has shown that an exclusive topical treatment fails to bring about complete understanding no matter how intensively the topic may have been studied or how much drill may have been given. Complete understanding develops slowly and cannot be forced. It calls not only for numerous experiences but also for a variety of experiences. For example, the fact that a pupil, after studying a topic, has written a test paper in which his work was 100 per cent right is by no means sufficient evidence that the topic has been mastered. The probability is that it merely has been remembered. Before real understanding can take place, a great deal of further assimilative experience is required. The pupil must be given work in future topics in which processes and principles previously taught continue to function. Thus, in learning to solve equations like $5x=15$, the pupils generally are required to work numerous exercises of this type. For many of them this is only the first step toward real understanding. They work mechanically. Each equation to be solved is a problem in itself. To generalize from the particular equations to the general form $ax=b$, they need further illustrations, as, for example, the circle formula, $2\pi r=c$; the percentage formula, $0.05b=p$; the interest formula $0.04p=i$; and

proportions, like $\frac{x}{2}=\frac{5}{8}$. Each offers a review and a new view of the equation. Complete understanding of $ax=b$ is the final outcome of all these experiences.

The topical plan of organizing materials creates the tendency of including under the various topics much that is irrelevant. Subject matter for which the pupil will have no immediate need in the course is brought in for no other purpose than to "round out" the topic. For example, when the pupil encounters a simple equation like $12x=105$, it might seem advisable to introduce the division axiom in the solution; but the topical plan would present not only the axiom that is needed in the solution but all the axioms used in solving other equations. Indeed, it often happens that the axioms are all presented long before any of them are actually needed. From the standpoint of the learner who fails to see the need and purpose of the work, this method presents the subject in a most uninteresting form.

The topical arrangement has developed courses in algebra which begin with lists of definitions and axioms, to be followed with chapters dealing with the operations, with the solution of equations, and finally with the problems to be solved by means of the equations. One of the ultimate aims of algebra is to develop a knowledge of the mechanics of algebra which will enable the pupil to solve problems; but when one examines the equations in the chapter on equations, they are found to be far more complex than the most difficult equations which arise in the solution of any of the problems the pupil is asked to solve. Likewise, one of the principal functions of the chapters on operations, fractions, and factoring is to prepare the pupil adequately to solve equations, because the solutions involve the fundamental operations, the manipulations of fractions, and factoring. However, most of the exercises used in teaching the fundamental operations, fractions, and factoring are of far greater complexity than those actually occurring in the solutions of the equations. The result is that some of the most abstract and most difficult phases of algebra are presented almost at the beginning of the course.

The logical organization of subject matter.—In arranging instructional materials in high-school mathematics, there has been an ex-

treme tendency to put together subject matter mainly because it belongs together "logically." This is to be expected in a course like demonstrative geometry, which has always been taught on account of the unusual opportunities for training in logical thinking; but it is also true of the other mathematical subjects.

In a strictly logical organization, psychological principles of learning are often sacrificed. Little attention is paid to the difficulties which pupils encounter in learning. With the exception of a few axioms and definitions, every new fact is to be proved. It matters little if the first principles studied are the most difficult as long as a logical sequence is observed. On the other hand, proofs are required even of the simplest principles which the pupil has known long before he came to the study of logical geometry.

When facts are arranged in logical sequence and when the order of difficulty is entirely disregarded, the learner repeatedly, and often very abruptly, passes from the most simple tasks to the most difficult, and conversely from the most difficult to the most simple. Hence the course lacks homogeneity and is unsatisfactory with secondary-school pupils. In the early courses in mathematics logical organization is less important than psychological organization.

A plan which collects principles and processes following each other in rapid succession, related only in the sense that a logical sequence is formed, breaks up the courses into isolated and unrelated details. It cannot lead to real understanding of mathematics no matter how logical the arrangement might be. Furthermore, when materials are arranged merely logically, associations between the facts and principles in a chapter are difficult to establish. Each lesson on a fact, principle, or process becomes a separate unit, taught and studied without clear connection with other facts, without definite insight as to how the facts are related to each other, and without understanding what each fact contributes to the whole. The goal of instruction and of study is ability of the pupil to recite on, or to discuss, each particular lesson taught, before taking up the study of the next lesson. Each fact, principle, or problem is to be learned, mastered, and tested separately. A successful recitation is considered sufficient evidence that a lesson has been learned. Tests consist of questions on isolated facts and problems rather than on a chapter as a whole.

The pupil may answer, satisfactorily, questions on each lesson without being able to recite on the topic and without being able to distinguish between the relative values of the parts of a topic. Comprehension of these relations and of the aims and purposes of the various processes, if attained at all, may or may not come as the pupil approaches the end of the chapter, but never at the beginning. Usually it is attempted to establish relations by way of summary and review, and only too frequently for no other purpose than preparation for a test.

The use of the project as a unifying factor of organization.—The discussion in the foregoing pages has presented several plans of organizing instructional materials. It should not be understood that they are the only available plans, but they have served the purposes of calling attention to some of the problems which arise in the organization of subject matter. Other plans have been designed which aim specifically to overcome some of the disadvantages that have been stated. One of them is known as the "project plan." As the name indicates, it attempts to make a project the unifying factor of organization and to eliminate artificial arrangement by presenting subject matter in a natural setting which makes a genuine appeal to the pupil.

The advantages and disadvantages of the project method of teaching have been discussed in chapter ii. There is much to say in favor of organizing mathematical materials around a central project, but the plan has its limitations. The most serious are the narrow scope of the mathematics that arise in connection with projects and the lack of definiteness and progressiveness which add so much to the values of the more systematically organized courses. The project has its place in such a course, but attempts to make it the sole basis of organization are sure to fail.

The unitary organization of mathematics.—Adults are frequently heard to remark that they recall less of the mathematics which they have studied in the high school than of any of the other subjects, although they always considered themselves good students of mathematics. The explanation may be found in the fact that they learned their specific lessons in mathematics daily, and were able, on the following day, to give a satisfactory report on each lesson, but that they failed to grasp the few really important

broad principles contained in courses in algebra or geometry. In fact, they mastered only assimilative materials, or applications of principles, and not the principles themselves; and they found it impossible to retain permanently this mass of materials.

It is a well-known fact that learning is more lasting if the pupil learns the whole rather than separate parts. Hence, materials should be put together that are economically and effectively learned together. Such a body of material must be presented to the pupil as a whole so as to give him a general conception of it as a "unit," in which such detailed facts and principles as have been selected are closely related to each other.

There are relatively few broad principles in high-school algebra and geometry. Teachers become acutely aware of this when trying to select materials for testing. Invariably the items are taken from certain "groups" of theorems in geometry or of processes in algebra. These groups are made up of facts and principles that are closely related to each other. They make good teaching units. The following are some of the typical illustrations of units in geometry: congruence of figures, similarity, proportionality, loci, circles, and areas. In algebra we have: polynomials, graphs, equations, signed numbers, formulas, and problems. Practically all of the materials that are usually presented in the traditional courses can be distributed among these units. Indeed, some belong to more than one unit. Thus, proportionality is related to similar figures, to circles, and to parallel lines. The decision as to where it should be placed depends largely on the question of where it is best presented, taught, and learned.

Furthermore, some of these units will contain too much material. They become unwieldy for teaching purposes and should therefore be divided into smaller units. For example, the study of the circle is too large a unit for secondary-school mathematics. It can be conveniently divided into the following units: relationships between angles, arcs, chords, and tangents; measurement of angles by means of arcs; proportional line segments in circles; and regular polygons related to circles.[1]

It is not the purpose of this chapter to discuss the problem of organizing specific mathematical units. That problem is being dis-

[1] E. R. Breslich, *Senior Mathematics*, Book II, chaps. ix–xii.

cussed in another volume.[1] It is desired, however, to make clear the meaning of the term "unit" as it is used in this book. To illustrate what is meant, we may consider in some detail the unit formed by a group of theorems in plane geometry which express relationships between angles and arcs of a circle.[2]

In a strictly logical organization the theorems would be studied separately, each having been assigned a place in a logical sequence. It is quite possible that a pupil may recite successfully and pass a satisfactory test on them but fail to see the connection between them and the relationships of each to a major topic which may be called the "measurement of angles by means of arcs." Because of the relationships among these theorems, they belong together pedagogically and should therefore be presented together and learned as a unit. The objectives to be attained in the unit call for more than mastery of the various theorems. They involve complete understanding of the general principle of measurement of angles by means of circle arcs.

As pointed out previously, one of the major characteristics of a unit is that it can be presented as a whole. It must be possible to give the pupil a clear view of the unit. This may be done as follows:

In your previous work in geometry you have had occasion to measure angles. You will recall that the unit of measure as a rule has been the "degree," although sometimes the right angle and the straight angle were used. Thus a given angle may be said to be 30°, one-third of a right angle, or one-sixth of a straight angle. You are now going to learn a new method of measuring angles, which, as you will recognize, is of considerable importance and value in solving certain problems and exercises. For example, it will be seen that angles A and B in the figure on page 143 are equal. For, it can be shown that it is possible to express their measures numerically in terms of one and the same arc, namely, CDE, on which both angles stand.

In a general way the problem of measurement presented in this unit may be stated in the following form: "Given an angle formed by two lines which intersect or touch a given circle; it is required to express the measure of the angle in terms of the arc, or arcs, cut off by its sides."

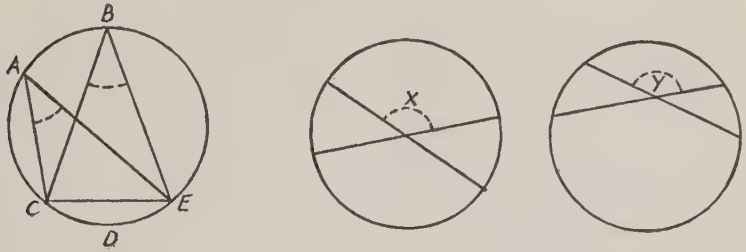
First you must know what is meant by the phrase "angle is measured by an arc." This is a brief way of saying that there are as many degrees of angle in the angle as there are degrees of arc in the arc.

You will readily see that the lines forming the angle may be placed in

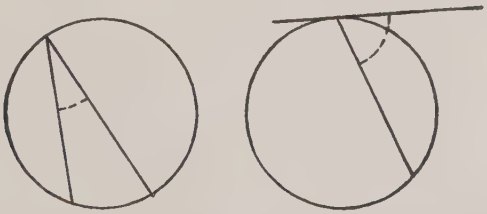
[1] *Administration of Mathematics in Secondary Schools.*

[2] *Senior Mathematics,* Book II, pp. 183–210.

various relations to the circle. Thus, the vertex of the angle may be inside of the circle. In that case it may either fall on the center, as the angle denoted by x; or not on the center, as the angle denoted by y. It will be shown that in both cases the angle is measured by one-half of the sum of the two arcs cut off by the angle and its vertical angle.



If the vertex of the angle lies on the circle, you have the two possibilities shown in the figures below. In the first of these figures, the sides of the angle are both chords; and in the second, one side is a chord and the other a tangent. It will be shown that the angle formed by the two lines is measured by one-half of the arc cut off, i.e., that the number of angle degrees is exactly half as large as the number of arc degrees in the arc cut off by the sides of the angle.

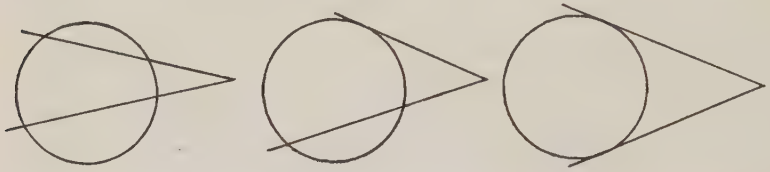


Finally, the vertex of the angle might fall outside of the circle, as in the three figures on page 144. It will be shown that in this case the angle is one-half of the difference between the arcs cut off.

Thus, the preview stated briefly and to the point, acquaints the pupil with the general principles to which this group of theorems is related. Even now he must feel that he has a fairly good conception of the unit. He knows the road he is going to travel when he turns to the textbook to study the detailed assimilative materials, i.e., theorems, exercises, and applications, which are to give him possession of the new method of measuring angles.

As he studies the separate theorems of the unit, he is constantly aware of their relation to each other and to the unit. Retention will be lasting because each contributes to the understanding of a broad principle.

Moreover, when theorems are grouped together that are illustrative of a general principle, the learner often has the advantage that the proofs of the various theorems resemble each other very closely. He will therefore study the unit with economy of work and time.



The test of understanding of the unit will be the successful use of the principle in a series of applications.

Summary of Chapter VI.—The chapter has shown the need of organizing instructional materials for purposes of effective teaching and learning. For the purpose of contrast the following types of organization were discussed.

1. The alphabetical arrangement. It is very effective for the purposes of reference but illustrates an extreme method of presenting facts without any attempt of relating them to each other.

2. The topical arrangement. The materials within each topic are brought into relation to each other, but the topics themselves are not. The disadvantages are that the pupil, trying to master each topic by itself before going on to the next, does not attain clear understanding of the subject as a whole. The method also develops a tendency to crowd into the topics materials which have little immediate value to the pupil.

3. The logical arrangement. It emphasizes training in logic. The method is not satisfactory, because psychological arrangement is more important in teaching pupils of high-school age than a rigorous logical sequence.

4. The project plan. It aims to relate the subject matter to projects of interest to the pupils but does not present a systematic development of the field of high-school mathematics.

5. The unitary organization. The following are the essential characteristics of a pedagogical unit as the term is used in this volume:

a) It must be a compact body of closely related facts and principles.

b) It must be possible to present the unit as a whole in concise form, giving the learner a clear, general conception of it.

c) It must be susceptible to testing for complete understanding.

The second characteristic implies that at the beginning of a unit a view of it should be so presented in a concise expository survey that the class will see clearly the road it is going to travel, getting a complete, even though only superficial, notion of the unit itself. Only the large principles are to be sketched. The preview must not go too much into detail and must not be too technical, but it should contribute much toward making thinking purposeful and learning permanent. The preview is followed by a period of assimilative study, during which the class absorbs and assimilates the principles taught, each pupil making them his own through the experiences and applications that are provided.

The third characteristic implies that at the end of a unit the class has not only understood each of the various principles taught, but that the pupils shall be able to give a good topical organization of it and write a satisfactory test on the unit.

The meaning of a pedagogical unit is ably discussed by Professor H. C. Morrison, of Chicago. He says:

> The course material which we find in the curriculum is valuable only as it is analyzed into significant units of learning. . . . . Much traditional school practice ignores the mastery of the true units of learning and in place thereof focuses its attention on the pupil's performance of assigned tasks.[1]

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[1] H. C. Morrison, *The Practice of Teaching in the Secondary School*, pp. 36–37.

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CHAPTER VII

TESTING AS A MEANS OF IMPROVING INSTRUCTION IN MATHEMATICS

The aims and purposes of testing.—In modern business every detail is examined critically. In the manufacture of an article every step in the production is studied carefully. There is no place for taking chances and for guessing. Errors are costly. Likewise in modern teaching, we cannot expect to determine the best and most effective teaching procedures by the method of trial and error. We must know the facts, and tests are the instruments used to obtain the desired facts.

Until recent date tests have been used almost entirely for the purpose of determining pupils' marks. In modern teaching, however, they have a much wider use. In fact, testing has become a definite part of the teaching and learning process. It will be recalled that a test may be used before the study of the unit to secure information regarding the pupil's experiential background. The preview at the beginning of the unit is followed by a test to identify the pupils who have, and who have not, understood what has been said. A test preceding the assignment shows whether the class is prepared to begin work on the assignment. Following the teaching of some topics or processes, practice tests are given. Finally, when the unit is completed, an examination is given to determine what the pupil has learned. Thus, the pupil's attainments are measured before, during, and after the study of the unit. Each test reflects the teacher's success or failure, and each serves a definite purpose. Marks may be determined from the various results; but the major purpose of each test is diagnosis, which in turn brings about improvement of instruction.

Tests in the teaching of mathematics may be classified according to the purposes for which they are designed. They are as follows:

1. Ability tests
2. Inventory tests

3. Practice tests
4. Standardized achievement tests
5. Unstandardized achievement tests
6. Standardized diagnostic tests
7. Unstandardized diagnostic tests
8. Research tests

Mathematical ability tests.—As the name indicates, the purpose of these tests is to determine the pupil's ability to study successfully secondary-school mathematics. The information obtained is used in various ways: (1) in advising pupils in regard to the election of courses in mathematics; (2) in classifying them in groups of different levels of ability for the purpose of teaching or of carrying on research; and (3) in analyzing special cases of pupils showing unusual ability, either high or low. When used for the purpose of preestimating the degree of success a pupil might be expected to have in special mathematical courses, they are called *prognostic tests.* The primary aim of prognostic tests is not to eliminate pupils by advising them against taking work in mathematics but to adapt mathematical instruction to their abilities.

The authors of ability tests have based their work upon careful analyses of the various mathematical subjects. They have tried to determine the specific abilities on which success in the study of mathematics depends, but it is evident that, although specific ability is an important factor in success, it is not the only one. General intelligence, industry, attitudes, home environment, and other factors are not to be overlooked. Furthermore, the tests have not yet reached the stage of perfection, and mistakes would be made even with the most refined instruments of measurement. Hence, care should be exercised in the application and use of the test results. When pupils are grouped according to ability, there should also be provision for correcting later any errors that have been made.

Test materials for measuring mathematical ability.—The following tests have been devised to measure the pupils' capacity to succeed in high-school mathematics:

Orleans, Joseph B., and Jacob S. *Geometry Prognostic Test.* The World Book Co., 1928.

———. *Algebra Prognostic Test.* New York: World Book Co., 1929.

Rogers, A. L. *Experimental Tests of Mathematical Ability and Their Prog-*

nostic Value. New York: Bureau of Publications, Teachers College, Columbia University, 1918.

Stoddard, G. D. *Iowa Placement Examinations, Mathematics Aptitude,* "Iowa Studies in Education," Vol. III, 1925.

Inventory tests.—This type of test has been discussed at length in chapter i. It has been found to be superior to the method of informal discussion for determining the pupil's background for the study of a course or of a unit in a course. If it is made sufficiently definite and comprehensive, it enables the teacher to tell in advance what knowledge of certain phases of the unit each individual possesses. It thereby simplifies the teaching problem. An example of an inventory test on a unit was found on pp. 6–7. The final test on a unit may easily be reduced to an inventory test for the unit by omitting all processes and facts with which the pupils rarely, if ever, have had any previous experiences. The making of the inventory test is therefore not different from that of the final test described on pp. 163–66. In secondary-school mathematics many pupils are handicapped because they lack proficiency in the arithmetical processes. The inventory test discovers quickly and in a most satisfactory manner deficiencies which it might take the teacher months to find. A test may therefore be constructed and administered which will measure skill in those arithmetical processes which occur in the study of a particular course in mathematics. On the basis of the results, remedial measures can then be taken to correct any deficiencies. An example of an inventory test in arithmetical computations and problem-solving is found in *Seventh-Year Mathematics.*[1] Samples of practice tests for remedial steps are found in the same textbook on pages 28 and 29 and in numerous other places of the book.

Practice tests.—In chapter iv attention was called to the importance and need of practice tests as a teaching device. A high degree of familiarity with some processes in mathematics is most desirable, and the teaching of these processes should therefore be followed immediately by practice tests. Modern textbooks provide such tests where they are needed.[2] Usually time limits are given for each test,

[1] Pages 1–11.

[2] E. R. Breslich, *Senior Mathematics,* Book I, pp. 258, 270, 278, 289, 301, and others.

not for the purpose of increasing speed but to give the pupil an idea as to what he might be expected to accomplish within a given time.

He can also use these exercises to test himself. Such practice will develop a new and friendly attitude toward all testing.

The traditional class examination.—The class examination has always been the most widely used form of testing. It is usually of the essay type, and it is administered at the end of a chapter, unit, or course. The major purposes of the class examination are to ascertain what the pupil has learned, to aid in the determination of the pupil's grade, and to decide questions of promotion or failure.

This type of examination is usually open to such criticisms as the following: The items are ordinarily made up hastily, the teacher selecting at random a few typical problems or questions. They are therefore apt to be too long or too short, and not at all comprehensive. They test memory and mechanical performance rather than understanding and ability to apply or to reason. They motivate aimless cramming instead of systematic review and organization. They are difficult to mark, and the teacher's grades are not accurate. It is next to impossible to grade a written examination twice in exactly the same way. Many uncontrolled factors, such as penmanship, personality of pupil, teacher, or even parent, and behavior of the pupil in class, influence the reader, and thereby the mark attached to the paper. Often the examination is totally different from what the pupils expected. They are disappointed and become discouraged. The tests have no diagnostic value and are therefore of no use in providing for remedial measures. The fact that the pupil has failed is of no help to the teacher unless he can tell where he has failed and the reason for his failure.

Notwithstanding the faults of the traditional class examination, the fact must not be overlooked that it is an instrument which, if improved and refined, could be used to supply much valuable information about the pupil. The foregoing criticisms do not prove that the examination should be abandoned, but indicate the lines along which improvement should be made.

The standardized achievement test.—The standardized tests have eliminated most of the faults of the class examination. They are made by experts, are scientifically designed, and make use of the most modern testing devices. During the process of construction

they have been tried out on many children to determine proper length and degree of difficulty. They are administered according to specific directions and under uniform time conditions. The values of the test items are clearly defined and determined by consensus of opinions of experts. The scoring is objective and eliminates the personal element. Norms have been established to stimulate the pupil to his greatest effort. The tests are published in more than one form and can be administered repeatedly to the same group. The results can be used to compare classes and schools with each other.

However, standardized achievement tests are not free from objections. They cover too large a field and cannot include everything that is essential, in which case they are not diagnostic. Usually they do not stress the most attractive phases of the course, and they mostly test information and mechanical performance rather than power and appreciation. They continue to use obsolete or questionable materials long after teachers have discarded them. The norms are said to lower effort rather than to raise it. Furthermore, as the courses improve, the standardized tests soon become out of date.

It is clear that the kind of testing whose major aims is to improve instruction must be done with an improved class examination which makes use of all the devices developed by the standardized tests but which is comprehensive enough to include all the essential facts, skills, and processes. It must determine what the pupil has learned, and indicate the remedial work to be done.

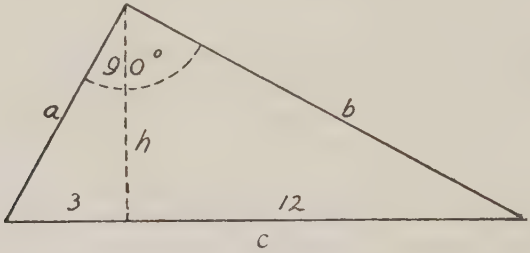
Objective tests.—In making an examination, the teacher must avail himself as much as possible of the so-called "brief-answer tests," e.g., the true-false test, completion test, choice-of-answer test, and matching test. The essay test is not to be discarded entirely, for it will always be very appropriate for certain types of materials. However, there is much to be said in favor of the brief-answer tests. Obviously, a wider field of subject matter can be covered by the pupil in a given time than is possible with an essay test. Since less writing is required, he will have more time to think. The brief-answer tests are objective and therefore easily scored, which means a saving of the teacher's time. The following examples illustrate objective tests.

A. *A completion test.*—Into each blank space of statements 1 and 2 below write the word which makes the sentence true.

1. If two lines intersect, opposite angles are _____ and adjacent lines are _____.

2. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b}=\frac{c+d}{d}$ by _____.

3. State in symbols three relations in one or several of the literal numbers a, b, c, and h, in the adjacent diagram which express three different principles. _____, _____, _____.



The completion test is particularly well adapted to test matters of information.

B. *A multiple-response test.*—Draw a line under the word that makes the following sentences true.

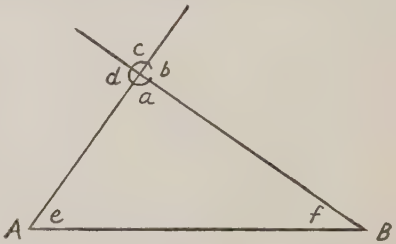
1. The converse of a theorem is true: always, sometimes, necessarily, never, generally.

2. If two parallel lines are cut by a transversal the alternate interior angles are always: supplementary, acute, equal, 90°, adjacent.

In using the multiple-response test, care must be taken that the responses suggested would be considered by the pupil as possible answers to the question.

C. *A matching test.*—Write into each blank space to the right of the following terms the best illustration shown in the diagram.

1. Acute angle _____
2. Vertex _____
3. Right angle _____
4. Triangle _____
5. Opposite angles _____
6. Adjacent angles _____



D. *A true-false test.*—If you feel that the results given in the following table are correct, place a check ($\surd$) mark in the column headed "True";

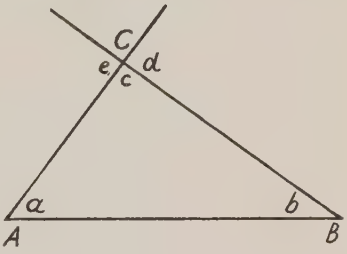
but if they are wrong, place a check mark in the column headed "False" and write the correct result in the next column.

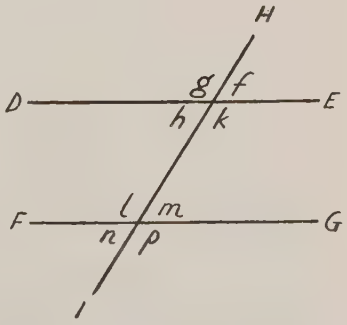
	Result	True	False	Correct Result
$\frac{2+5}{3+5}$	$\frac{2}{3}$			
$ax+bx$	abx			
$\frac{m^2+m}{m^2}$	$\frac{m+1}{m}$			

A complete final examination for the unit on angles and angle relations is given below.[1] The unit aims to develop understanding of the following: new words used in the study of angles; a classification of angles; several fundamental constructions; and a certain type of equation used in the unit. The test follows:

I. Using the notation of the adjoined figures, give the best illustration for each of the following terms:

1. Adjacent angles_____
2. Parallel lines_____
3. Right angle_____
4. Oblique lines_____
5. Transversal_____
6. Supplementary angles _____
7. Corresponding angles _____
8. Opposite angles_____
9. Exterior angle of a triangle _____
10. Perpendicular lines_____
11. Acute angle_____
12. Alternate interior angles _____
13. Complementary angles_____
14. Obtuse angle_____





[1] *Ibid.*, Book I, pp. 122–24.

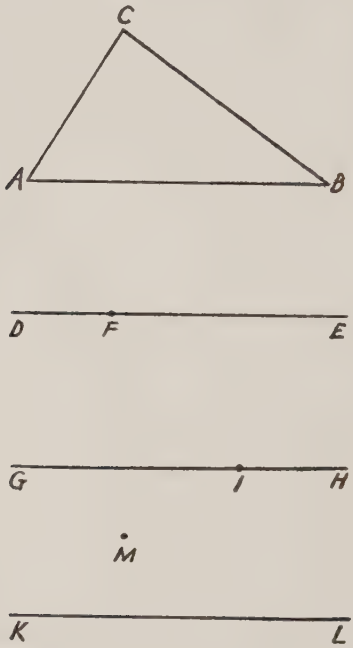
II. In the following incomplete statements write into each of the blank spaces the word needed to make the sentence complete:

1. A right angle is equal to ____ degrees.
2. The sum of the angles of a triangle is ____ degrees.
3. The acute angles of a right triangle are _____ to each other.
4. The adjacent angles formed by two lines perpendicular to each other are _____ and _____.
5. Supplements of equal angles are _____.
6. If two sides of a triangle are equal, the triangle is _____.

III. Solve the following equations:

1. $5x-10+x+2=15$ 2. $5x+9=2x+20$ 3. $\frac{x+2}{3}+\frac{2x-1}{3}=18$

IV. In triangle ABC measure angle B. *Result:* $\angle B=$____°.



V. Make the constructions called for with ruler and compasses, showing all construction arcs:

1. Bisect $\angle B$.
2. On DE at F draw an angle equal at $\angle A$.
3. At I construct a line perpendicular to GH.
4. From M construct a line perpendicular to KL.

VI. Express in symbols the following verbal statements:

	Symbolic Forms
1. x and y are complementary.	1.
2. a and b are supplementary.	2.
3. The sum of four angles just covering the plane around a point is 360°.	3.
4. a is greater than b.	4.

VII. For each of the following problems state the equation, but do not solve.

	Equations
1. x is the supplement of $x+20$. Find x.	1.
2. One of two complementary angles is 18° larger than the other. Find each angle.	2.
3. The first angle of a triangle is $\frac{3}{4}$ as large as the second. The third is 3 times as large as the first. Find the angles.	3.

VIII. For each of the following statements put a cross (×) in the column headed "True" if you believe that it is true. If you think that the

statement is false, put a cross in the column headed "False" and write the correct statement to the right of the cross.

	True	False	Correct Statement
1. $13x+11=24x$.	1.		
2. $9x+(2x+4)=11x+4$.	2.		
3. 50° and 40° are complementary.	3.		
4. If $4x=16$, then $x=12$.	4.		
5. If $x=3$, then $5x+2=25$.	5.		
6. If $x+1=6$, then $x+3=18$.	6.		

Objective tests for proofs in geometry.—In the traditional examination the teacher of demonstrative geometry usually selects three or four theorems to be proved because the class period does not allow time to give proofs of all of the theorems in a given unit. The degree of success with the selected proofs is then taken as a measure of the pupil's knowledge of the unit. The fallacy of this assumption is apparent. Hence, there is real need for comprehensive tests on theorems in geometry.

The selection of the theorems to be tested does not need to be arbitrary. The teacher may follow the suggestions of the National Committee[1] and of the College Entrance Examination Board. They have recommended a limited number of basic theorems as the minimum with which every pupil of geometry should become thoroughly familiar. The number of basic theorems in a given unit is usually small, and the pupil's understanding can be tested in many cases without requiring complete proofs. The following suggestions simplify the problem of making tests on proofs of geometric theorems:

1. Let the pupil draw only the diagram and state in symbols what is given and what is to be proved. This may be used when the proof of a theorem is simple and when experience has shown that pupils have difficulty in stating the hypothesis and conclusion.

2. Give only the statements of the proof but let the pupil supply

[1] A report of the National Committee on Mathematical Requirements: *The Reorganization of Mathematics in Secondary Education* (1923), pp. 57–58.

the reasons. This device should be used when the proof is very long but otherwise simple.

3. Give the statements of a proof in disarranged order and let the pupil arrange them in logical sequence. This device should be employed in difficult proofs.

4. Give the statements in logical sequence, and also a mixed list of reasons. Let the pupil then select the proper reason for each statement.

5. Let the pupil make a complete proof from a mixed list of statements and reasons.

The following is a complete test for a unit in demonstrative geometry.[1] It will be noted that no complete proofs are called for. Tests on complete proofs and on original exercises should be given from time to time during the study of the unit.

To save time and to eliminate all unnecessary writing, the statements and reasons in the test are numbered, and the pupil writes only the numbers.

Teachers who have tried the foregoing devices to test units of demonstrative geometry have sometimes found that pupils can write a complete proof in less time than it takes to arrange the statements or reasons. Likewise, they can write the whole proof in less time than it takes them to prove just a reasoning cycle taken out of the proof. The explanation is that those pupils have memorized the proof and can give it best in the form in which it was memorized. Thus, what seems to be an objection to the brief answer test is really an argument for it, because it will tend to discourage rote memory in logical demonstrations.

TEST ON SIMILAR FIGURES. THEOREM OF PYTHAGORAS

A. For each of the following cases state the theorem which makes the triangles similar:

1. $\triangle$I $\sim$ $\triangle$II. Theorem:_____

_____

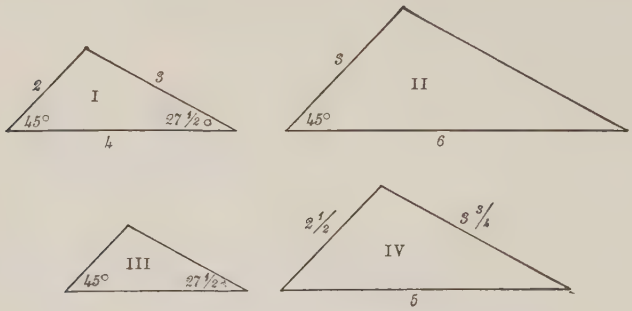
2. $\triangle$I $\sim$ $\triangle$III. Theorem:_____

_____

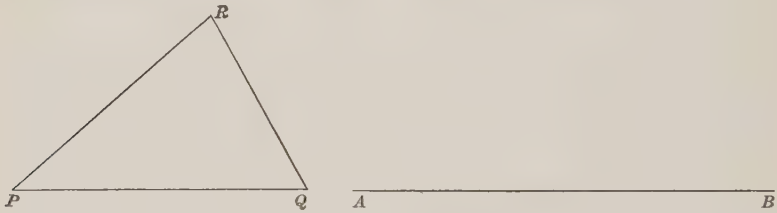
[1] E. R. Breslich, *Mathematical Achievement Test VI* (Chicago: University of Chicago Press).

3. $\triangle$I $\sim$ $\triangle$IV. Theorem: _____

_____



B. On AB as one side construct a triangle similar to $\triangle PQR$.



C. Make the following statements complete:

If $\frac{a}{b}=\frac{b}{c}$, _____ is the _____

between a and c. _____ is the third _____ to a and b.

D. On the line below construct x so that

$\frac{2}{x}=\frac{x}{5}$

E. Solve: (1) $\frac{3}{x}=\frac{x}{12}$ (2) $3m^2=6-7m$

F. Simplify: (1) $\sqrt{180}$ (2) $\sqrt{x^3y^2}$ (3) $\sqrt{128a^2b^3}$

G. 1. Write the equation for finding the height of a chimney whose shadow is 42 yards when the shadow of a 6 foot vertical pole is 5.5 feet long. Equation: _____

2. Write the equations for finding the sides of a triangle whose perimeter is 18 yards if the sides of a similar triangle are 5.2 yards, 6.5 yards, and 7.9 yards. Equations:

_____

_____

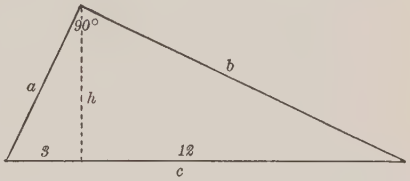
_____

H. For the adjoined diagram state in symbols three relations involving some of the letters a, b, c, h, but no two relations should express the same principle.

1. _____

2. _____

3. _____



I. For the following theorem draw the diagram, state in symbols what is given and what is to be proved, and state how the diagram is drawn: *Two triangles are similar if the ratio of two sides of one is equal to the ratio of two sides of the other, and if the angles included between these sides are equal.*

Diagram

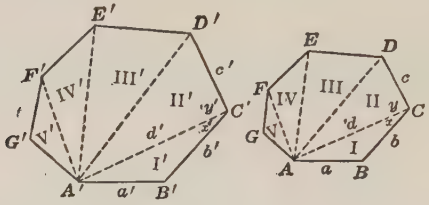
Given: _____

To prove: _____

Explain diagram: _____

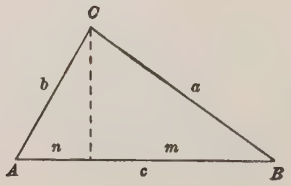
_____

J. The statements below are sufficient to prove $\triangle I \sim \triangle I'$ and $\triangle II \sim \triangle II'$. Arrange them in logical order and write the numbers of the ordered statements in the blank spaces.



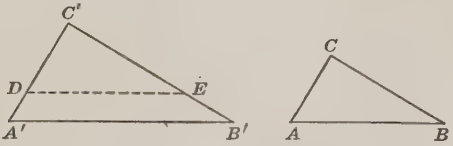
	Statements	*Ordered Statements*
1.	$\frac{b}{b'}=\frac{d}{d'}$	1. _____
2.	$x=x'$	2. _____
3.	$\angle B=\angle B'$	3. _____
4.	$\triangle II \sim \triangle II'$	4. _____
5.	$\frac{a}{a'}=\frac{b}{b'}$	5. _____
6.	$\frac{b}{b'}=\frac{c}{c'}$	6. _____
7.	$\triangle I \sim \triangle I'$	7. _____
8.	$y=y'$	8. _____
9.	$\frac{c}{c'}=\frac{d}{d'}$	9. _____
10.	$C=C'$	10. _____

K. In the following proof of the theorem of Pythagoras state the authority for each step.



Statements	*Authorities*
1. $\frac{m}{a}=\frac{a}{c}$	1, 2. _____
2. $\frac{n}{b}=\frac{b}{c}$	_____
3. $a^2=mc$	3, 4. _____
4. $b^2=nc$	_____
5. $a^2+b^2=(m+n)c$	5. _____
6. $a^2+b^2=c^2$	6. _____

L. The following statements prove that *two triangles are similar if the corresponding sides are in proportion.* From the authorities below choose for each statement the one that belongs to it and write its number in the blank space to the right.



Statements	*Authorities*
1. $\frac{CA}{C'A'}=\frac{CB}{C'B'}$	1. _____
2. $\frac{C'D}{C'A'}=\frac{C'E}{C'B'}$	2. _____
3. $\triangle DEC' \backsim \triangle A'B'C'$	3. _____
4. $\frac{C'D}{C'A'}=\frac{DE}{A'B'}$	4. _____
5. $\frac{CA}{C'A'}=\frac{AB}{A'B'}$	5. _____
6. $\frac{C'D}{C'A'}=\frac{AB}{A'B'}$	6. _____
7. $\frac{DE}{A'B'}=\frac{AB}{A'B'}$	7. _____

8. $DE = AB$ 8. _____

9. $\triangle DEC' \cong \triangle ABC$ 9. _____

10. $\triangle ABC \sim \triangle A'B'C'$ 10. _____

Authorities:

1. Corresponding sides of similar triangles are in proportion.
2. Given.
3. By substitution.
4. If equals are multiplied by the same number, the products are equal
5. Two triangles are similar if two sides of one are proportional to two sides of the other and if the angles included between these sides are equal.
6. Things equal to the same thing are equal to each other.
7. Two triangles are congruent if the sides of one are equal to the sides of the other.

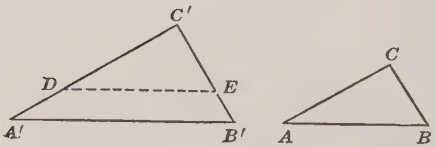
M. The following statements and authorities prove that *two triangles are similar if two angles of one are equal to two angles of the other.* Arrange the statements and authorities in logical sequence and enter the numbers of the arranged statements and authorities on the blanks below.

Statements

1. $\triangle DEC' \cong \triangle ABC$
2. $\angle A = \angle A'$
3. $\triangle ABC \sim \triangle A'B'C'$
4. $C'D = CA$
5. $DE \| A'B'$
6. $\angle D = \angle A'$
7. $C'E = CB$
8. $\triangle DEC' \sim \triangle A'B'C'$
9. $\angle C = \angle C'$
10. $\angle D = \angle A$

Authorities

1. Things equal to the same thing are equal to each other.
2. Given.
3. By construction.
4. By substitution.
5. The corresponding parts of congruent triangles are equal.
6. A line parallel to a side of a triangle forms with the other two sides a triangle similar to the given triangle.
7. Two triangles are congruent if two sides and the included angle are equal respectively.
8. Two lines are parallel if the corresponding angles formed with a transversal are equal.



	Statements	*Authorities*
Proof	1. _____	1. _____
	2. _____	2. _____
	3. _____	3. _____
	4. _____	4. _____
	5. _____	5. _____
	6. _____	6. _____
	7. _____	7. _____
	8. _____	8. _____
	9. _____	9. _____
	10. _____	10. _____

The improved class examination.—Let it be assumed that a unit has been finished and that throughout the study of the unit the teacher has employed the various types of tests that help secure objective evidence that understanding has been attained. The class is then ready for the final testing to determine whether the objectives of the unit have been attained and whether the essentials of the unit have actually been acquired, to discover weaknesses if they exist, and to take remedial measures if necessary.

The authors of several textbooks have recognized the need for tests on the various chapters or units of instruction and have included them in their books.[1] If no published tests are available, the teachers will have to prepare their own unit tests. Obviously, such examinations must be constructed with unusual care, and teachers will have to study and acquire the art of designing tests. At first this may take a considerable amount of time; but after the tests are made, teachers will be more than repaid for their efforts because the tests can be used over and over. In the following pages it will be shown how to construct the new type of class examination.

It is a good plan to make the final test on a unit before the unit is taken up rather than to wait until it is finished. The teacher will then be in a position to present the unit in well-organized form, to select the best uses and applications, and to place particular em-

[1] See *Mathematical Achievement Tests*, I–XI for *Senior Mathematics*, Grade X (Chicago: University of Chicago Press).

phasis on important phases of the work during the study of the unit. Teaching will thereby become purposeful and intelligent. Not only will those facts and principles be taught that are of greatest permanent value, but an appreciation of such values will be developed among the pupils.

The first step in making the test is to decide on definite objectives of the unit (chap. viii). Then the instructional materials of the unit are examined in detail and classified. All that is essential and to be thoroughly emphasized in teaching must be plainly identified. For, new concepts and facts are to be learned, study habits to be acquired, definitions to be derived, principles to be developed, processes to be mastered, procedures to be understood, skills to be attained, and applications to be made. Furthermore, certain attitudes and appreciations are to be developed. When the unit is finished, the test must reveal whether the objectives have been fully attained.

Every unit contains material that is not to be tested, such as historical notes, illustrative matter, certain applications brought in merely to show the usefulness of mathematics, material presented to motivate the work, and facts introduced only to give the pupil a first view of a new process or topic. For example, let us suppose that among the instructional materials of the unit on straight lines the teacher finds a historical sketch of the development of the metric system, some uses of lines in picturing percentages, and several problems involving drawings of polygons.[1] Since the major purpose of the historical sketch is to arouse interest and to develop appreciation of the metric system, it would be a mistake to include it in the test. Details of materials intended to make the work attractive to the pupil might easily be changed into drudgery when they were intended to be a pleasure.

Since the materials involving polygons and percentages are to present only a first view of these topics, and since they are more thoroughly discussed in later units, they are not to be used as test material until later in the course.

On the other hand, the study of lines involves a considerable amount of arithmetical work in connection with measuring and comparing line segments. Since proficiency in arithmetic is con-

[1] *Seventh-Year Mathematics*, chap. i.

sidered important for successful work in later units, the book provides a thorough review of the arithmetical processes which all pupils must master. Hence, items involving these processes are suitable test material.

Another aim of the unit is to develop skill in the use of ruler and squared paper. Items to test these skills should therefore be included.

Illustrations of items for a unit test.—In the following, several examples will be presented to suggest suitable test questions:[1]

A. A test on graphs.—Three types of graphs are usually taught in an introductory chapter: the bar graph, the line graph, and the graph of a precise mathematical law. When the pupil has finished the chapter, he is expected to be able (1) to make graphs and (2) to interpret what they mean.

In making a graph, the following activities are involved:

1. If a precise law is to be represented, a table of corresponding values of the variables has to be made.

2. If the numbers in the table are very large, they must be rounded off. Thus 3,502,614 is rounded off to 3.5. A common fraction must usually be changed to a decimal fraction. Thus $8\frac{2}{3}$ is changed to 8.7.

3. The axes are to be drawn and suitable units selected. If the numbers are large, the unit must be small; and conversely, if the numbers are small, a large unit may be used.

4. The number pairs must be plotted.

5. The points are then joined by a line, if a line graph is being made.

Thus, besides testing the pupil's ability of making and interpreting graphs, provision must be made separately for steps (1) to (5) to offer opportunities for all possible errors in order that a diagnosis can be made of errors and difficulties that should call for reteaching.

A complete test on graphs should include items like the following:

a) Using 1,000 as a unit, express as thousands or tenths of thousands the numbers: 8,756; 3,594; 1,867; 391; 587.

b) Make a table of corresponding values of the variables d and t in the equation $d=3.5t$.

[1] Taken from *Senior Mathematics* (University of Chicago Press), Book I, pp. 41–42.

c) The following number pairs are to be represented graphically:

1	2	3	4	5	6
18	31	26	45	12	32

Draw the reference axes and mark off convenient units.

d) Plot the number pairs

2	3.8
5	1.6
3	2.4

e) Make a bar graph representing John's earnings for the week:

Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
$0.85	0.72	0.96	1.10	0.43	1.80

Interpret the graph.

f) Make a line graph representing the following temperature readings:

9 A.M.	10 A.M.	11 A.M.	12 M.	1 P.M.	2 P.M.	3 P.M.
75°	78°	84°	85°	85°	82°	80°

Interpret the graph.

g) Make a graph of the formula $c=6.2r$.

B. A test on problem-solving.—There are two important steps to be tested: (1) that of deriving the equation and (2) that of solving the equation. It is evident that each of them can be further subdivided. For example, deriving the equation involves reading and comprehending the statement of the problem; understanding the situation described in the problem; selecting the given and required numerical facts; expressing the facts in symbols; and choosing the right relationships. Solving the equation involves combining similar terms; multiplying each term by same number to remove the denominators; adding to, or subtracting from, both sides of the equation; and dividing both sides of the equation by a number. Textbooks provide practice in all of these steps and processes. Usually they are presented in the chapters preceding the units on problems or equations. They have therefore been tested separately as the various chapters were finished. During the study of the units

on solving problems or equations, further practice in these processes should be given because they occur in situations that are new to the learner. However, it is not necessary to include in the final examination items to test these processes.

The following represent three typical test items on solving problems:[1]

1. Write the equation for the following problem but do not solve: A farmer wishes to divide a 550-acre farm among three sons, giving to the youngest son twice as many acres as to the oldest, and to the second son as many as to the oldest. How many acres should each receive?

*Equation:*_____

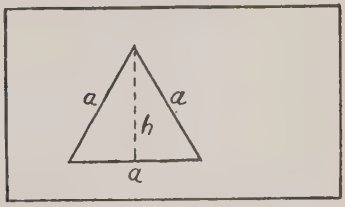
2. Solve and check the following equation: $x+8+2x=24-5x$.

3. Solve the following problem: What sum of money placed at 5 per cent simple interest for two years will amount to $348?

Giving the test a preliminary trial.—The next step in making the final examination of a unit is to have the test items typed or mimeographed. They should then be administered to a small representative group of pupils of various levels of ability. The purpose is to formulate and try out instructions for giving the tests, to discover deficiencies, and to determine approximate time limits. Often a question which is entirely clear to the teacher is ambiguous to the pupil. Undesirable items are eliminated, and poorly formed statements are re-worded. The test is then repeatedly administered and further revised until the teacher is satisfied with it.

Misfires are apt to occur from a number of causes. The following examples illustrate some of them:

1. The statement "The area of a _____ is equal to the product of the base by the altitude" may refer to the rectangle, the parallelogram, or both. The question should be re-worded so as to make only the desired answer possible.



2. The statement "Express the area of an equilateral triangle in the symbols of the adjacent figure" is ambiguous and might be answered by pupils in several ways: $\frac{a^2}{4}\sqrt{3}$, $\frac{1}{2}ah$, impossible, or all three. It therefore needs re-wording.

[1] *Senior Mathematics*, Book I, pp. 225–26.

3. The statement "Measure two sides of a triangle and find the ratio" calls for measuring, knowing the meaning of ratio, and dividing. If all three are to be tested, ignorance of the first destroys the possibility of securing evidence on the second and third. The statement should therefore be supplemented by three others calling separately for measuring, for the meaning of ratio, and for dividing. In general, when an item of a test depends on the correct response to another item, additional questions are needed to test both separately. Thus, a question calling for the solution of a verbal problem should be supplemented by others, such as deriving an equation and solving an equation.

Formulating instructions for the administration of the test.—When the test has reached a form in which it is entirely satisfactory to the teacher, instructions should be formulated to make it self-administering. The following procedure of administering the examination has proved to be very effective:

The pupils' desks are cleared of all unnecessary materials. Each pupil sees to it that he has the necessary paper, pencil, compasses, protractor, eraser, etc. Classroom conditions must be made as nearly normal as possible. The tests are then distributed by one or several pupils, and work begins. If the test is too long for one class period, it should be divided in halves to be given on two successive days. For those pupils who finish ahead of the class, an advance assignment should be made to keep them profitably employed throughout the class period. The other pupils are allowed ample time to show what they really know and to finish the test without being unduly hurried. When a pupil has finished, the teacher should take up his paper quietly and without any comment.

Scoring the test.—A test should always be scored by the teacher. He will note general difficulties which require further teaching. Answers should be marked either right or wrong by means of convenient symbols, such as a check (✓) mark for right and a cross (×) for wrong. It frequently happens that a pupil understands the principles to be tested but makes mistakes due to carelessness in details or to poor preparation in former work. For example, he may reduce an algebraic fraction correctly in every detail but make a mistake in copying or in an arithmetical computation. His failure is caused by poor habits of study. Such items may be marked with a check within a circle Ⓥ. These marks will readily identify the careless or superficial pupils.

The crosses on the test paper and the check marks within the circles are then tabulated on a score sheet similar to the one shown below. The number right are totaled horizontally and vertically and a "U" or "S" is entered in the last column, according as the paper is unsatisfactory or satisfactory.

SCORE SHEET

Mathematics_____ Test_____ Unit_____

Period_____ Teacher_____ Date_____

Test Items / Class Enrolment	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	Total Right	Satisfactory
1																						
2																						
3																						
4																						
5																						
6																						
7																						
8																						
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Total Right																						

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Diagnosis and reteaching.—Just as the inventory test shows the pupil's background and enables the teacher to make the right start in the teaching of the unit, so the final test suggests remedial work where it is needed, and identifies pupils who may be exempted from further work. The totals at the bottom of the score sheet show which items are not understood and on which the whole class needs more teaching. After reteaching the work represented by these items, a sufficient amount of practice work is given and a second test is administered. If the results are not yet satisfactory, the procedure is repeated.

The totals in the right column identify the pupils who need reteaching on the work not retaught to the whole class. This is done in private conference with the teacher. With the test paper before them, teacher and pupil examine and discuss the errors and deficiencies. When the difficulty has been located and explained, practice work is given and later a second individual test is administered. During the process of reteaching and retesting of pupils, the whole class goes on with the study of the new unit.

Pupils who write perfect test papers may be excused from all reteaching and retesting and spend their time on special assignments or on voluntary projects.

The testing program outlined above can be used to secure objective information regarding difficulties which pupils in general encounter in the study of mathematics. It helps the teacher to identify the problems, processes, and concepts which pupils repeatedly fail to master, and to discover defects, strengths, and weaknesses of the course. It is therefore an effective aid to the teacher in improving his method of instruction.

Testing for the purpose of research.—Tests are the instruments with which to determine strengths and weaknesses of the course and relative difficulties of various types of subject matter. They enable us to construct a curriculum scientifically. The following diagnostic procedure may be employed: Every problem or exercise contained in a test is carefully analyzed and resolved into its component elements, steps, and skills. The steps are then scored separately by assigning a score of 1 to each process performed correctly. The number of correct responses by the whole group of pupils is

then divided by the number of pupils in the group, and thus an index of attainment for a particular step is obtained. Similarly, the ratio of the total score made by the group on the complete problem to the total possible score is a measure of attainment for the whole problem in question. Thus, if a certain step in an exercise is done correctly by 76 pupils in a group of 80, the attainment is denoted by the ratio $\frac{76}{80}$, or 0.95.

A key for the separate processes and problems to be scored should accompany every test. The following illustrations are typical of the method used in scoring:

1. *Multiplication of monomials.*—This is one of the simplest processes in algebra. It is also one of the fundamental operations, and must therefore be mastered by all pupils. It is not sufficient to know that a pupil has failed to work the problem correctly: the teacher must know where in the problem the difficulty lies. Otherwise, to assign practice work in multiplying monomials might do more harm than good. Practice must be given in the particular step or steps which caused the failure.

The problem involves three steps: the determination of the sign, the arithmetical product, and the literal product. Each of these is scored separately.

For example, in $(-12m^5)\ (-16mr^3)$,

the sign is +	1
the arithmetical product is 192	1
and the literal product is m^6r^3	1

Therefore, the score for a correct response to the problem is 3.

It must be kept in mind that the score does not indicate a measure of the degree of difficulty or the value of the step, but simply states that the three steps in the process have been performed correctly.

Diagnosis is not always easy to make. A pupil might fail to have the arithmetical product correct because he does not know his arithmetic or because the other two steps in the problem distracted and monopolized his attention. In the first case he needs practice in arithmetic; in the second case he needs more practice with multiplying monomials in which arithmetic is involved.

2. *Division of monomials.*—This process also consists of three steps. Thus in $(-6a^2cx^2y) \div (2axy)$,

the sign is −	1
the arithmetical quotient is 3	1
the literal quotient is acx	1

Therefore, the score for a correct solution of the problem is 3.

3. *Multiplication of a polynomial by a monomial.*—The difference between this problem and Problem 1 is that it involves the multiplication of a number of monomials. Therefore, the score is 1, meaning that the operation of multiplying monomials has been performed correctly when it occurs in the multiplication of a polynomial by a monomial.

Thus, $(3a^3-2a^2+5a)\ (-2a^2) = -6a^5+4a^4-10a^3$.

Mistakes may be caused either by ignorance of the process of multiplying monomials or by inability to perform the process correctly in a more complex situation.

4. *Multiplication of polynomials.*—The steps in this process are: (1) the multiplication of polynomials by monomials and (2) the addition of the resulting polynomials. Accordingly, the product $(a^3-a^2-1)\ (a+1)$ is scored as follows:

a^4-a^3-a		
a^3-a^2-1	Multiplication of a polynomial by monomial	1
$a^4 \quad -a^2-a-1$	Addition of polynomials	1

Therefore, the score for the problem is 2.

5. *Division of polynomials.*—The following steps are involved in long division:

a^4-2a^2+1	a^2-1	Arranging terms in ascending or descending order	1
	a^2-1	Division of monomials	1
$a^4-\ a^2$		Multiplication of polynomials by monomials	1
$-\ a^2+1$		Subtraction of polynomials	1
$-\ a^2+1$			
Check:			1

Therefore, the score for the problem is 5.

6. *Solution of a linear equation.*—In solving the equation $y-\frac{2y-4}{3}=\frac{2(5-y)+7}{9}$, the following steps have to be taken:

a) Multiplying each term by 9 to remove the denominators of the fractions:

$9y-3(2y-4)=2(5-y)+7$ 1

b) Multiplying polynomials by monomials:

$9y-6y+12=10-2y+7$ 1

c) Combining similar terms:

$3y+12=17-2y$ 1

d) Adding to, and subtracting from, both members:

$5y=5$ 1

e) Dividing both members:

$y=1$ 1

f) Checking the original equation:

$$1+\tfrac{2}{3}=\frac{8+7}{9} \quad \ldots\ldots$$

$1\tfrac{2}{3}=1\tfrac{2}{3}$ 1

Therefore, the score for the problem is 6.

The analysis of the processes of long division and of solving equations with fractions throws some light on the difficulty of both. They involve a number of fundamental operations and provide the best type of training in them. Mastery of a process is not attained, unless the pupil can carry out the process correctly when it occurs in such complex situations as the foregoing illustrate.

7. *Solution of a quadratic equation.*—In solving $4x^2+x-39=0$ by the factoring method, the following steps have to be taken:

$(4x+13)(x-3)=0$. The left member is factored 1

$4x+13=0$, $x-3=0$ } Two linear equations are derived 1

$x=-1\tfrac{3}{4}$, $x=3$ } The two equations are solved 1

The solutions are checked 1

Therefore, the score for the problem is 4.

8. *Solution of simultaneous equations.*—Two methods of solution are analyzed:

a) Solution by graph:

(1) The tables are made 1

(2) The axes are drawn and the units selected 1

(3) The points are plotted..... 1
(4) The graphs are drawn..... 1
(5) The solution is found from the point of intersection..... 1

Therefore, the score for the problem is 5.

b) Solution by addition or subtraction:

$$\begin{array}{rr} & 3m+7n=\ 34 \\ & 7m+8n=\ 46 \\ \hline \text{Multiply:} & 21m+49n=238 \\ & 21m+24n=138 \\ \hline \end{array} \quad \ldots\ldots 1$$

Subtract: $25n=100$..... 1
Solve: $n=\ 4$..... 1
Substitute: $3m+28=\ 34$..... 1
Solve: $\begin{cases} 3m=\ 6 \\ m=\ 2 \end{cases}$..... 1
State the solution $\therefore (m, n)=\ (2, 4)$..... 1

Therefore, the score for the problem is 6.

9. *Solution of verbal problems.*—The following two examples illustrate the method of analyzing verbal problems:

a) A messenger who travels 6 miles an hour is followed after 2 hours by a second who travels $7\frac{1}{2}$ miles an hour. In how many hours will the second overtake the first? The following steps are involved in solving the problem:

Solution:

	Time	Rate	Distance	
(1) The data are stated for one of the messengers:	x	6	$6x$	1
(2) The data are stated for the second messenger:	$x-2$	$7\frac{1}{2}$	$7\frac{1}{2}(x-2)$	1

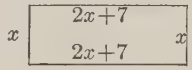
(3) The equation is derived: $6x=7\frac{1}{2}(x-2)$..... 1
(4) The equation is solved: $\begin{cases} 6x=\dfrac{15x-30}{2} \\ x=10 \end{cases}$..... 1

The score for the problem is 4.

b) A rectangle is to be made so that one side is 7 feet longer than twice the other, the perimeter being 54 feet. Make a sketch of the rectangle and find the length of the sides.

Solution:

(1) Express the data of the problem in symbols.
Let x be the number of feet in one side.
Then $2x+7$ is the number of feet in the other side..... 1
Draw the figure:



(2) State the equation:
$2x+4x+14=54$..... 1

(3) Solve the equation { Combine: $6x+14=54$..... / Subtract: $6x=40$..... / Divide: $x=6\frac{2}{3}$..... } 1

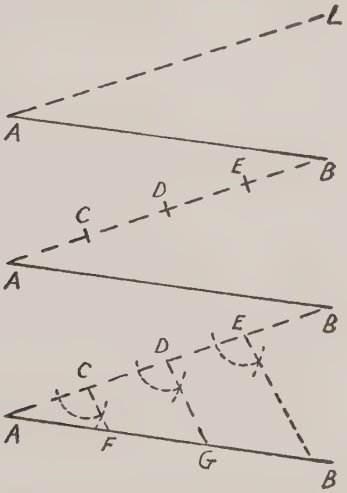
(4) The number of feet in the second side is:
$2x+7=20\frac{1}{3}$..... 1

(5) Check:
$6\frac{2}{3}+6\frac{2}{3}+20\frac{1}{3}+20\frac{1}{3}=54$..... 1

Therefore, the score for the problem is 5.

10. *A geometric construction with ruler and compasses.*—The following example illustrates how to analyze a geometric construction.

Problem: To divide a segment into three equal parts.



Given: A segment AB.....
Required: To divide AB into three equal parts } 1

Construction:

Draw the helping line AL 1
Lay off three equal segments on AL 1
Draw the parallels to EB 1

Proof: $AC=CD=DE$ 1
$CF||DG||EB$ 1
$AF=FG=GB$ 1

The score for the problem is 7.

The study of the results of a particular test.—The application of the preceding discussion will be illustrated with a detailed consideration of one test which was given at the end of the unit on the fundamental operations of algebra. The test and key for scoring follow:

THE TEST ITEMS

I. Change to the simplest form by collecting terms:
1. $6x^2+8-3x^2-5-9x^2-7$
2. $3g+f-8g-6f-7g-5f$
3. $-8t^3-4t+6t^2-3t+2t^3-9t$
4. $-2ab-3g+7ab+7g+6ab+4g$

II. 1. If $c=4$ and $f=2$, what does $2c^3-3f$ equal?
2. If $a=3$ and $b=2$, what does $3ab+ab^2$ equal?
3. If $x=3$ and $y=4$, what does xy^2-2xy equal?
4. If $r=2$ and $s=4$, what does r^2+3r^2s equal?

III. 1. From $2d+13f$ take $7d+14f-6g$.
2. Take $9h+14k$ from $7h-3k+8s$.
3. Subtract $6m-11n+13p$ from $m+3n+7p$.

IV. Perform the indicated operations and reduce to the simplest form:
1. $3(7y)$
2. $(2a)\ (4ab^2)$
3. $\frac{2}{3}$ of $9m$
4. $(4x)\ (-3xy^3)$
5. $(a^3)\ (-3a)\ (-2a)$
6. $(-3xy^3)^4$

V. Simplify: $7(5x+9)$; $5(8x-4)$; $-6(5x+2)$;
$-9(4x-6)$; $7(-6x-4)$; $-8(-4x-6)$.

VI. Perform the indicated operations and reduce to the simplest form:

1. $\frac{12n}{4}$
2. $6c^3 \div 2c^3$
3. $\frac{-8a^2b}{4a^2}$
4. $\frac{4x^4}{5} \div 2x^2$
5. $\frac{7a}{15} \div \frac{7a^2}{20}$
6. $\frac{-12x^2y^2 \cdot (x-2)}{-3x^2y^2}$

VII. Perform the indicated operations and reduce to the simplest form:

1. $(2a^2+7a-9)\ (5a-1)$
2. $\frac{18m^2n-27mn^2}{9mn}$
3. $(x^3-x^2-4x+4) \div (x^2-3x+2)$

VIII. Show graphically each of the following statements to be true. Explain your drawing.

1. $(8)+(-2)=6$
2. $(-6)-(-2)=-4$
3. $(-2)\ (-4)=8$
4. $ab+ac=a(b+c)$

KEY FOR THE TEST

		Total Score
I. $-6x^2-4$. Combining similar terms	2	8
II. $(2)\ (64)-(3)\ (2)=122$. Substituting, multiplying, combining	3	12
III. $-5d-f+6g$. Each term is scored 1, if right	3	9
IV. $21y$. Numerical product	1	13
$8a^2b^2$. Numerical product, literal product	2	
$6m$. Numerical product	1	
$-12x^2y^3$. Sign, numerical product, literal product	3	
$+6a^5$	3	
$+81x^4y^{12}$	3	
V. $35x+63$. Each term scores 1, if correct	2	12
VI. $3n$. Arithmetical quotient	1	15
$3c$. Numerical part, literal part	2	
$-2b$. Sign, numerical part, literal part	3	
$\frac{2}{5}x^2$. Invert, multiply, reduce	3	
$\frac{4}{3a}$. Invert, multiply, reduce	3	
$+4(x-2)$. Sign, numerical part, literal part	3	

VII. $10a^3+35a^2-45a$. Product of polynomial by monomial
$-2a^2-7a+9$. Product of polynomial by monomial 1

$10a^3+33a^2-52a+9$. Combining terms 1

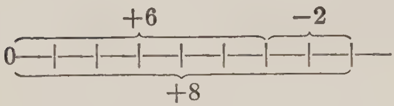
$\dfrac{9mn\,(2m-3n)}{9mn}$ Factor, reduce 2

$$\begin{array}{l|l} x^3-x^2-4x+4 & x^2-3x+2 \\ \hline & x+2 \\ x^3-3x^2+2x & \\ \hline \quad 2x^2-6x+4 & \\ \quad 2x^2-6x+4 & \\ \hline \end{array}$$

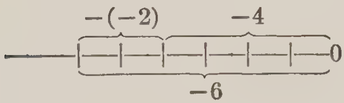
Divide 1
Multiply 1
Subtract 1

Total: 7

VIII.



. . . . 3



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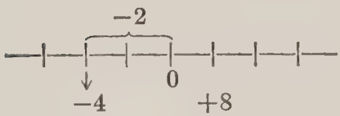

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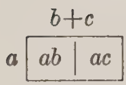

$ab+ac$, a, $b+c$ 3

Total: 12

Eighty-eight distinct operations had to be performed in this test. The achievement quotients were computed by dividing the score actually made by the possible score. The processes were then arranged in order of achievement, as in the table on page 179.

The table shows that the simplest process was the multiplication of a binomial by a monomial and that the most difficult processes were the addition and subtraction of polynomials when they occur in multiplication and division of polynomials.

The table reveals that the results attained with a process, when tested by itself, are not the same as when the process has to be performed in different situations. For example, in multiplying mono-

COMPARISON OF ACHIEVEMENTS IN SEPARATE PROCESSES WITH ACHIEVEMENTS IN THE SAME PROCESSES WHEN THEY ARISE IN VARIOUS SITUATIONS

Process		Achievement
1. Multiplication of a binomial by a monomial		93
2. Graphical addition of positive and negative numbers		92
3. Combination of similar terms		84
4. Subtraction of polynomials		84
5. Multiplication of monomials		83
Sign	80	
Numerical product	86	
Literal product	80	
6. Reduction of fractions		78
Sign	72	
Numerical part	80	
Literal part	78	
7. Multiplication of two polynomials		73
Product of monomials and polynomials	80	
Adding polynomials	50	
8. Evaluation		69
Substitution	72	
Product of positive and negative numbers	69	
Combination of terms	67	
9. Multiplying positive and negative numbers graphically		65
10. Dividing by a fraction		61
Inversion	67	
Multiplication	62	
Reduction	55	
11. Graphical subtraction		61
12. Division of polynomials		56
Division of monomials	64	
Product of polynomial by monomial	54	
Subtraction of polynomials	50	

mials (process 5) the group shows an attainment ratio of 83. In process 7 it drops down to 80; in process 8, to 69; and in process 10, to 62, although the test items in process 5 were more difficult than the items in processes 7, 8, and 10.

Similarly, for reductions of fractions the ratio is 78, but in process 10 it drops to 55.

For combining terms the ratio is 84, but in evaluation problems it drops to 67.

For multiplication of polynomials by monomials the ratio is 80 in process 7, and 54 in process 12.

For subtraction of polynomials the ratio is 84 for the process alone; but in division it is 50.

Invariably when a new process is taught, the old processes that are involved offer new difficulties and deserve new attention. The tests show again and again that when a process is taught separately, it is not mastered even if a class has attained the point of perfection. The attainment ratio drops when the process is met in new situations. From the standpoint of teaching, it is an important discovery that separate practice on isolated processes does not seem to secure real and lasting understanding, and that mastery is not attained until the pupil has had repeated experiences with a process in a large number and variety of situations and has given evidence of understanding with test items involving such situations.

After the difficulties in the test are located, the analysis may be carried one step farther by means of conferences with the teachers of the various classes. The following table gives the attainment ratios for every problem for each of four classes and for the total group. It shows wide variation in the results attained by the classes, the largest interval being 32 and the smallest 11. The table should be read as follows: Class I, in problem 1, had an attainment of 96; etc. It raises the question why the largest variation in the various classes is in Problem 1 and the smallest in Problem 5.

Attainment Ratios of Four Classes for the Test

Problems	1	2	3	4	5	6	7	8
Class I	96	85	94	94	98	88	78	81
Class II	83	71	85	79	98	64	64	59
Class III	64	86	67	75	87	66	56	.....
Class IV	92	69	77	81	90	71	56	62
All classes	88	71	84	86	96	83	64	69
Variations	32	17	27	19	11	24	22	22

Class I makes the best showing, and Class III the poorest. An explanation of this difference was obtained by consulting the teachers and the records of the classes.

The problems were next ranked in order of attainment, and presented in the table below, which shows that the variations in rank are small. Thus, Problem 5 was easy for all classes and Problem 2 was hard. The remaining problems very nearly kept their positions. The table shows a uniformity of the four classes in the results of the teaching, since the ranks of the problems are practically the same although the classes are clearly not of equal ability.

DIFFICULTY RANKS OF PROBLEMS SOLVED BY FOUR CLASSES

Problems	1	2	3	4	5	6	7	8
Ranks for Class I	2	6	3	3	1	5	8	7
Ranks for Class II	3	5	[2]	4	1	6	6	8
Ranks for Class III	[6]	[2]	4	3	1	5	7	8
Ranks for Class IV	1	6	4	3	2	5	8	7
Ranks for all classes	2	6	4	3	1	5	8	7

Of special interest are the three exceptions marked with squares. When the teacher of Class II was asked why his class did so well with Problem 3, he said that former experience had taught him that subtraction of polynomials is a difficult process and that he always placed special emphasis on this operation in his teaching. He apparently accomplished good results, because in the subtraction of polynomials which enters also in Problem 7, as part of the long-division process, his class was superior to all others, the ratios for this process being respectively 54 per cent, 74 per cent, 38 per cent, and 39 per cent. This teacher's method of presenting problems of the type of Problem 3 was therefore carefully studied by the other instructors.

Class III did not succeed well with Problem 1. The teacher's explanation was that the word "collecting" was new to the pupils, the word "combining" having been used in class. Sometimes the use of an unfamiliar word in a test confuses the pupils, and the results therefore do not give evidence whether the pupils are really able to work the problem. However, evidence of their inability to solve this problem may be found in the fact that the pupils were not any more successful when the same operation occurred elsewhere. For example, terms are to be collected in the third step in Problem 7. The ratios for this step for the various classes were 81 per cent,

53 per cent, 23 per cent, and 60 per cent, showing that Class III was deficient in the process even when no instructions as to "collecting" or "combining" were given. Finally, a further examination of the papers of the pupils verified the fact that they did not understand the process, for such mistakes as $a^2+a^2=a^4$ were frequent.

On the basis of the results of this study the teacher during the next class hour retaught the process, and similar errors were carefully corrected in the later part of the course.

That Class III can be trained, with careful teaching, to do as good work as the others, or even better work, is seen from the fact that they surpassed all other classes in Problem 2.

Summary.—The chapter has emphasized the importance of testing as an integral part of the teaching-and-learning process. The major purpose of all testing is improvement of instruction.

Mathematical-ability tests may be used for purposes of grouping pupils into homogeneous classes and of identifying those who have unusual high or low ability.

The inventory test gives the teacher valuable information about the experiential background of the class.

The standardized achievement test enables him to judge the results of instruction by comparison of classes and schools.

The improved class examination, however, is of greatest value as an instrument for diagnosing and reteaching. The following suggestions have been given for making the examination:

1. Employ brief-answer test items.
2. Make the test before the unit is taken up for study.
3. Decide upon the objectives of the unit.
4. Examine the instructional materials and select the essentials which pupils are expected to master.
5. Omit from the test all materials intended to be merely illustrative, to make the work interesting, and to motivate the unit.
6. Omit also materials which are presented in the unit for the first time, only for the purpose of giving the pupil a first view.
7. Examine each test item that has been retained and make sure that it is clearly stated and unambiguous.
8. In case of a complex process, determine the component parts and provide separate test items for each.

Tests should be carefully scored and the results tabulated. They

are then to be followed by remedial work, and further testing if necessary.

The improved class examination, on account of its diagnostic value, is an excellent instrument for research purposes. Much is to be learned about the course that will contribute further to the improvement of instruction.

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CHAPTER VIII

THE AIMS OF TEACHING SECONDARY-SCHOOL MATHEMATICS

The teacher needs to familiarize himself with the aims of mathematics.—The preceding chapters have been concerned with discussions of the activities which the teacher of mathematics must learn to perform well if he expects to be successful, and with the devices and procedures that will enable him to accomplish results in the most effective manner. But little has been said about the instructional materials to be taught. In general, the teacher is given no choice in selecting and organizing these materials. That problem is usually worked out for him by others who are trained and competent in this type of work. As a rule he is given an adopted textbook and a course of study which he is expected to follow. His job is to adapt the work to the needs and interests of the pupil and the community, and to choose the most effective activities, teaching procedures, and devices which will assist the pupil in assimilating the instructional materials. To be sure, the time may come, if he studies the problem, when, he, too, will be called upon to make his contribution to the curriculum. He will then need to familiarize himself with the principles on which the selection and organization of subject matter are based. It is not the purpose of this chapter to discuss or present the detailed facts, items, processes, etc., which make up the course of study, but merely to present a list of aims and purposes in the teaching of mathematics.

The aims and purposes of the teaching of mathematics are closely related to the selection of materials for teaching purposes. Unless the teacher gives serious thought to the aims of his subject, his work will lack real purpose. He will be teaching just bare facts. Authors of textbooks seem to be mostly concerned about the selection, organization, and presentation of materials. Hence, the textbooks do not give much information about the purposes of instructional materials. Apparently, it is left to the teacher to decide upon

the goals of instruction and to find out why the subject matter presented in the textbook is worth while.

The first step in the solution of this problem is to acquire definite knowledge of the aims of education and of the mathematics course as a whole. With these objectives before him the teacher may then proceed to determine definitely the aims of the various units to be taught and then the instructional materials which contribute to the unit aims. The following are illustrations of general aims to be attained in most units:

1. Real understandings of the meanings of a definite number of concepts, processes, principles, and relationships
2. Development of valuable skills, such as computations in arithmetic, factoring in algebra, and using the instruments in geometry
3. Ability to solve problems of the same degree of difficulty and of the same types as are presented in the unit, and power to use mathematical modes of thinking
4. Appreciation of the importance and values of the materials contained in the unit and of the unit itself
5. Development of certain desirable attitudes toward the unit, leading to interest in mathematics and a desire for further study
6. Development of effective habits of study

To illustrate how the instructional materials of a given unit may be made to contribute to the unit aims, we shall use as an example the teaching of "measurement of line segments." This properly belongs to the unit on the straight line, which aims to develop a clear conception and complete understanding of the nature and properties of straight lines. Measurement is one phase of that unit, and it contributes the following items to the unit:

1. Understandings:
   Knowledge of the wide use of lines in diagrams, designs, graphs, and surveying
   How lines are drawn to scale to enable the pupil to study and to solve problems
   Acquaintance with a variety of units of length
   Knowledge of how the units of length have been developed
   How straight lines are used to indicate directions or distances
   How the edge of a ruler is divided into tenths or sixteenths
   Understanding of the meanings of physical line, geometric line, length of a line, ratio of lines, average length, scale drawing, metric system, exact and approximate measure

2. Skills:
   Skill in the use of the ruler, compasses, and squared paper in measurement
   Skill in estimating lengths
3. Problems and methods:
   Finding unknown lengths from given or known facts
   Changing from one system of measures to another
   Direct measurement with instruments
   Indirect measurement by use of such geometric principles as similarity, and congruence, by trigonometric ratios and by such algebraic relationships as $a^2+b^2=c^2$
4. Appreciations: Appreciating the facts that—
   Measurement is widely used
   Measurement is always approximate
   The degree of accuracy in measuring determines the degree of accuracy in lengths derived from these measures
   It is important to be able to estimate lengths of lines without the use of instruments
5. Attitudes: Interest in—
   Problems of finding lengths of lines
   The historical development of units of length
   Further study of the topic
6. Habits:
   To do neat work in making drawings
   To be accurate in measuring with instruments
   To pre-estimate results
   To observe geometric forms and to recognize mathematical facts and principles

It is readily seen that some of the aims mentioned above are to be kept in mind in practically every unit of the course and may therefore be considered course or subject aims as well as unit aims. Furthermore, some course aims are common to several or all courses and thereby become general mathematical aims. Finally many of the general mathematical aims are also general aims of education. Thus, the usefulness of some of the facts, principles, processes, habits, and skills taught within a given unit extends far beyond the unit. For example, measuring line segments with a metric ruler contributes to a fuller understanding of the decimal notation taught in arithmetic, and finding the ratios of two measured line segments contributes to the mastery of the

process of dividing by a decimal fraction. A thorough acquaintance with the aims of arithmetic broadens the view of the teacher who is presenting the unit on line segments. The teacher who is aware that both understandings are aims of the teaching of arithmetic will not fail to give them the emphasis which they deserve.

Similarly, complete understanding of length as derived from measurement of a segment contributes to certain understandings to be acquired in geometry, trigonometry, and analytic geometry. Thus, to understand fully the meaning of the concept "sine ratio," the student must have a clear understanding of what is meant by the ratio of two line segments, which in turn presupposes an understanding of the length of a single segment.

Habits of neatness and accuracy, understanding of relationships, and appreciation of the degree of precision in measurement are fundamental to success in all mathematics. They are therefore classified among the general aims of mathematics.

Finally, knowledge of what is meant by a standard unit, mastery of the process of division, appreciation of the degree of accuracy in measurements, and skill in the use of the mathematical instruments are essential in vocational work. Hence, these genuine mathematical aims may help the pupil to be better prepared for his future vocation.

The teacher who is thoroughly familiar with the general educational aims will have no difficulty in making evident connections between the aims of the course, or subject taught and the general objectives. He must see the general aims clearly and keep them constantly in mind in his teaching. Otherwise he will fail to attain the fullest values of his subject.

The general objectives of secondary education.—In the effort of making his subject contribute to the general educational objectives the teacher encounters serious difficulties because at present there is no single statement of objectives on which all educators unanimously agree. The formulation of the real objectives is still in a stage of development. The tendency seems to be for each curriculum specialist to set up a system of objectives of his own.

One of the first to offer a solution of the problem of deriving definite sound principles for curriculum construction for modern education was Herbert Spencer. He divided the ultimate aims of

education into three classes: economic efficiency, domestic efficiency, and civic-social efficiency. Complete living is the end to be achieved, and subject matter and methods must be chosen with reference to this end. He presents a list of life-activities to attain the ends of education. They are as follows:[1]

1. Activities which directly minister to self-preservation
2. Activities which, by securing the necessities of life, indirectly minister to self-preservation
3. Activities which have for their end the rearing and discipline of offspring
4. Activities which are involved in the maintenance of proper social and political relations
5. Activities which make up the leisure part of life, devoted to the gratification of the tastes and feelings

According to Spencer, any subject would be judged according to the extent to which it introduces the pupil into these activities.

Bobbitt[2] comes to the conclusion that the abilities, attitudes, habits, appreciations, and forms of knowledge that men need are the objectives of the curriculum, and that pupil-activities are to be planned in the various subjects so as to contribute to the attainment of the general objectives. Education should prepare definitely to perform the activities of adult life. These activities are classified in ten groups:

1. Language activities
2. Health activities
3. Citizenship activities
4. General social activities
5. Spare-time activities
6. General mental activities
7. Religious activities
8. Parental activities
9. Non-vocational practical activities
10. Vocational activities

The most commonly adopted formulation of the objectives of education has been made by a commission appointed by the National Education Association. This commission, known as the "Commission on the Reorganization of Secondary Education," has

[1] Herbert Spencer, *Education: Intellectual, Moral, and Physical.*

[2] Franklin Bobbitt, *How to Make a Curriculum.*

laid down the following seven goals of secondary education, known as the "Cardinal Principles of Secondary Education."[1]

1. Health of individual and of community
2. Command of the fundamental processes
3. Worthy home membership
4. Vocational guidance and preparation
5. Citizenship in a democracy
6. Worthy use of leisure time
7. Ethical character

With these and other systems of objectives before him, the teacher may adopt the following procedure:

1. To study the various systems and select the objectives toward which he feels his subject has contributions to make
2. To identify definite instructional materials which contribute to the selected objectives and which should receive corresponding emphasis in teaching
3. To select the pupil activities through which the objectives are to be attained
4. To choose the most effective teaching procedures and devices to attain the general objectives

This foregoing procedure does not attempt to determine the subject objectives upon the basis of the objectives of education, but it aims to make the subjects contribute to the fullest extent to the realization of the educational objectives. Thus, if the teacher accepts the "Seven Cardinal Principles" as the most suitable formulation, his next problem will be to determine the contributions his subject is able to make.

This procedure has been employed by the Subcommittee on Junior High School Mathematics on the Committee on Standards for Use in the Reorganization of Secondary School Curricula of the North Central Association.[2] The subcommittee uses the technique set up by the Committee on Standards, which selected four ultimate objectives: (1) the health objective, "to secure and maintain a condition of personal good health and physical fitness"; (2) the leisure-time objective, "to use leisure time in right ways"; (3) the

[1] "Cardinal Principles of Education," *Bureau of Education, Bulletin No. 35.*

[2] R. Schorling, "Report of the Subcommittee on Junior High School Mathematics," *The North Central Association Quarterly*, II (March, 1928), 9–32.

social-ultimate objective, "to sustain successfully social relationships, such as civic, domestic, community, and the like"; (4) the vocational objective, "to engage successfully in exploratory and vocational activities."

Apparently the committee has selected four of the "Cardinal Principles," namely, health, worthy use of leisure, citizenship, and vocation. The remaining three have not been left out, for the committee has also set up certain "Immediate Objectives" for purposes of accomplishing the ultimate objectives in classroom work. These are as follows: (1) acquiring fruitful knowledge, (2) developing interests, motives, ideals, attitudes, and appreciations, (3) development of mental techniques in memory, imagination, judgment and reasoning, and (4) acquiring right habits of conduct and useful skills in living. It is seen that "command of fundamental processes," one of the "Cardinal Principles," might be easily included under "acquiring useful skills in living." Similarly, "ethical character" and "worthy home membership" are so closely related to the ultimate objectives that they may easily be combined with some of them.

In regard to these objectives a word of warning is in order. The various subjects taught in the secondary schools cannot be expected to contribute to each of the general objectives equally and to the same extent. For example, it must be evident that mathematics has a real contribution to make to the vocational and social objectives. Even a superficial glance at the list of objectives submitted at the end of this chapter will show that there is an abundance of subject matter which is directly useful to all pupils. On the other hand, contributions to the health objective must necessarily be few in mathematics. They are better cared for in other subjects. In mathematics they must be limited almost entirely to health data which may be employed in connection with the teaching of graphs; to interpretations of tables of heights and weights; to measurements of food values and problems relating to them; to computations of a lighting surface in a room or the breathing space to be allowed each person; and to a few other applications. Any attempt to go much farther will lead to waste from the standpoint of the mathematics to be taught and will make no real contributions to the health objective.

Teachers of mathematics are rarely qualified to discuss the nature of diseases, the care of patients, or laws of health. In attempting to do this, they will slight the real objectives, which are the mathematical objectives of the teaching of graphs, namely, the acquisition of skill in the making and interpretation of graphs and a full understanding of the graphical method of procedure. To accomplish this aim, the teacher will need a large variety of instructional materials among which such health data as can be easily understood and appreciated by pupils should be included, but no excessive attention should be paid to health data just because health happens to be one of the general objectives. One of the best contributions of mathematics to the health objective is to develop a thorough knowledge of arithmetic, because without it the individual will find it difficult to understand many discussions relating to health.

The same caution should be exercised in regard to the leisure-time objectives. In the anxiety of making a good showing of values to be derived from the study of mathematics, teachers are apt to crowd into their courses materials which actually interfere with the accomplishment of the real purposes of the subject. For example, interest in puzzles in newspapers and in shallow recreational materials as a leisure-time objective need not be cultivated in classes in mathematics. Too many people waste valuable time on it. What is needed is to develop genuine interest in the work done in the classroom and to create in the minds of the pupils a desire to continue such work outside of the classroom. For example, when a genuine taste for the beauty of designs or an appreciation of geometric forms in art and nature has been aroused, pupils will be stimulated to go to the mathematical library to read more about it in books and magazines. Some will voluntarily undertake independent projects. Leisure time spent on work of this type is well employed, develops study habits, and creates the kind of enjoyment which is experienced by adults engaged in research. The teacher who succeeds in developing interest in such work has indeed made a contribution to the leisure-time objective, because his pupils are learning to make the most of the hours of leisure at their disposal.

The National Committee's views on mathematical aims and values.—It has been customary in discussions of general aims and values of

secondary-school mathematics to classify them in three groups: the practical, the disciplinary, and the cultural values. The National Committee on Mathematical Requirements accepts that classification with the modification that it should be understood that in a broad sense the truly disciplinary and cultural aims are also practical, i.e., that the three classes are not mutually exclusive.[1] The following is an outline of the aims presented by the committee.

I. Practical aims
   1. The immediate and undisputed utility of the fundamental processes of arithmetic
      a) A progressive increase of understanding of the nature of the fundamental operations and power to apply them in new situations
      b) Exercise of common sense and judgment in computing from approximate data, familiarity with the effect of small errors in measurement, the determination of figures to be used in computing and to be retained in the result
      c) The development of self-reliance in the handling of numerical problems through the consistent use of checks
   2. An understanding of the language of algebra
   3. A study of the fundamental laws of algebra
   4. The ability to understand and interpret correctly graphic representations
   5. Familiarity with the geometric forms common in nature, industry, and life; mensuration of these forms; development of space perception; exercise of spatial imagination

It appears that the general mathematical practical aims listed by the committee are really selected from the practical aims of the separate subjects, i.e., of arithmetic, algebra, and geometry.

II. Disciplinary aims
   1. The acquisition in precise form of the ideas or concepts in terms of which the quantitative thinking of the world is done
   2. Development of the ability to think clearly in terms of such ideas and concepts. This involves training in analysis of a complex situation, recognition of logical relations, and generalization
   3. Acquisition of mental habits and attitudes
   4. The idea of relationship and dependence

[1] Report of the National Committee on Mathematical Requirements, *The Reorganization of Mathematics in Secondary Education*, pp. 6–12.

III. Cultural aims

1. Acquisition of appreciation of beauty in geometrical forms
2. Ideals of perfection as to a logical structure, precision of statement and thought, logical reasoning, discrimination between true and false
3. Appreciation of the power of mathematics

As to the relative importance of the various general aims, the committee states its attitudes as follows:

The practical aims enumerated above, in spite of their vital importance, may, without danger, be given a secondary position in seeking to formulate the general point of view which should govern the teacher, provided that they receive due recognition in the selection of material and that the necessary minimum of technical drill is insisted upon.

The primary purpose of the teaching of mathematics should be to develop those powers of understanding and of analyzing relations of quantity and of space which are necessary to an insight into and control over our environment and to an appreciation of the progress of civilization in its various aspects, and to develop those habits of thought and of action which will make these powers effective in the life of the individual.

Schultze on values and aims of mathematical teaching.[1]—Schultze divides the aims and values into three groups: the practical value, the disciplinary value, and minor functions.

Among the practical values he emphasizes the use of mathematics in science, the influence of mathematics upon life, and the value of mathematics to the individual.

The disciplinary value of mathematics lies most of all in the exercise of reasoning powers. Mathematics is especially fitted for this purpose on account of its simplicity, accuracy, certainty of results, originality, similarity to the reasoning of life, and amount of reasoning.

Among the minor advantages of mathematical study he states: development of power of concentration and of constructive imagination, growth of self-reliance, development of character, increased ability to use English correctly, and increase in general culture.

[1] Arthur Schultze, *The Teaching of Mathematics*, pp. 15–29. New York: Macmillan Co., 1923.

Young on values of mathematics.[1]—Young uses three classes of aims: the practical value, mathematics as a mode of thought, and other functions of mathematics.

I. The practical values are: the intimate connection of the subject with everyday life, its use in occupations, its informational value, mathematics in nature.

II. The chief value of mathematics, according to Young, is the fact that it exemplifies certain modes of thought, such as grasping a situation and drawing conclusions.

III. Other values are:

1. Generalizing conceptions
2. Information and use of symbolic language
3. The finished form in treatment of topics
4. Early discoveries
5. Knowledge for its own sake
6. Cultivation of reverence for truth
7. Cultivating the habit of self-scrutiny
8. The aesthetic side of mathematics
9. Development of imagination
10. Cultivating power of attention
11. Fostering habits of neatness

Schorling on objectives.[2]—Schorling breaks away from the traditional classification of aims by grouping the objectives of mathematics in four ways, viz., attitudes, concepts, information, and skills or abilities. He compiled a list of 520 statements or facts which he termed "objectives," "elements," or "items." They were derived from 29 objective studies, 7 junior high school texts, 2 studies determining practice from courses of study, and the outline of the National Committee on Mathematical Requirements. They were then submitted to two groups of "competent judges," of ten people. The items accepted on the basis of any three of the five criteria were classified and included among the four groups. This gave 9 attitudes, 63 concepts, 127 abilities, and 106 items of information. Most of the remaining items were grouped in the same

[1] J. W. A. Young, *The Teaching of Mathematics*, pp. 9–52. New York: Longmans, Green & Co., 1924.

[2] Raleigh Schorling, *A Tentative List of Objectives in the Teaching of Junior High School Mathemathics.* Ann Arbor: George Wahr, 1925.

way as "subsidiary items." The subsidiary lists contain 18 attitudes, 19 concepts, 46 abilities, and 66 items of information.

In applying the foregoing results in classroom instruction, the teacher is apt to become confused. Schorling himself calls attention to one obvious defect of this study, saying: "The writer does not accept the basic list throughout. In fact, there are a few things in the subsidiary list that he believes far more important than some things in the basic list." The basic list of 305 elements is worth studying. It presents definite materials to be taught; but in drawing conclusions as to the minimum essentials of a course or as to the relative values of the items in the two lists, the teacher must exercise considerable care. Relative values have not been established by the method employed in the study.

Furthermore, the reader in search of objectives will be confused by the variety of the terminology used. Some of the items listed are clearly general mathematical objectives, others are subject objectives, and most of them are merely important assimilative material which are to contribute to the attainment of the objectives. Thus, four of the nine attitudes are general mathematical objectives, namely: assuming responsibility for correct results; recognizing importance of an "estimated answer" as a control in the computation of a problem; being interested in, and constantly scrutinizing, personal growth in the skills of mathematics; and being critical of the extent to which a computed result based on measurement is accurate. On the other hand, appreciating the use of geometry in art and appreciating the use of geometry in architecture are objectives of geometry. However, appreciating necessity of desirable habits of thrift, being critical of investments, and appreciating the power of compound interest, are non-mathematical objectives which usually are to be attained in courses in arithmetic.

Items 94 to 103, expressing ability to use the formulas $A = bh$, $A = x^2$, $A = \pi r^2$, $c = \pi d$, $v = lwh$, etc., are merely assimilative materials contributing to three unit objectives listed as items 105, 106, and 107, namely: the abilities to solve any formula for any letter; to evaluate any formula for any letter; and to translate a practical rule of procedure in arithmetic into a formula. Other objectives of the unit on formulas, such as appreciation of the value and useful-

ness of the formula and a thorough understanding of the meaning of the formula, have not been listed at all.

Not mastery of assimilative materials, but mastery of the unit objectives, should be the outcome of the study of a unit. Thus, throughout the unit on graphs the teacher must keep in mind that the objectives of that unit are not the temperature graphs, health graphs, and others, but understanding, appreciation, and interpretation of graphical representation, ability to represent numerical facts graphically, and ability to represent formulas graphically. In factoring, the real objectives are not mastery of the various cases, but number insight and ability to combine and decompose algebraic number expressions. Mastery of the factoring of such individual cases as a^2-b^2 does not alone lead to these objectives. For example, the pupil may be able to factor a^2-b^2 and yet fail to see that it is but a special case of ax^2+bx+c. To be sure, the greatest care must be exercised in the choice of the assimilative materials but it is of major importance that definite unit objectives are set up and that the teachers have the unit objectives constantly in mind as they present the assimilative materials of the unit.

General mathematical aims.—In the foregoing pages it has been asserted that the subject of mathematics cannot afford to stand outside of the general system of education. It must help bear the burden of the pupil's education and contribute its share to the attainment of the objectives of education. Indeed, it is expected not only to take its place among the other subjects but to accomplish some things better than some of the other subjects.

Leaders in the teaching of mathematics have always been studying the problem of aims and values of mathematics. Much has been written on the topic in the educational mathematical literature. For, *the value the student derives from the study depends largely on the way it is taught.* We cannot expect mathematics to develop logical, precise, and careful thinking if the pupil is allowed to be consistently illogical and careless in his work. If the study teaches him only isolated facts, his chances for developing growth in power of generalization are small. The values of mathematics are not attained by accident. The teacher must have in mind definite purposes in his teaching. It is quite possible for a pupil to master each exercise in a given list separately without understanding the gen-

eral principle which all the exercises illustrate. However, with the right kind of teaching, each exercise can be made to contribute to the understanding of the general principle, and after a sufficient number of exercises the principle will be thoroughly assimilated. It has happened that pupils have solved a long set of equations of the form $ax=b$, have written perfect test papers on such equations, and yet failed to understand the real meaning of the equation $ax=b$. For, later when they met the equation $2\pi r=24$, they declared emphatically that they never saw such an equation. Hence, they were unable to solve it. The training received with the former exercises did not carry over because no real understanding was attained the first time. In fact, it might take many other experiences and a larger variety of situations before the pupil will be able to recognize any equation of the form $ax=b$.

Unless the processes of algebra are actually mastered, i.e., unless the teacher of algebra has evidence that the pupil recognizes and performs them correctly in any given situation, the pupil will be unable to perform them when they are encountered outside of the classroom in mathematics, e.g., in problems in physics. The practical values of the subject are therefore lost to him.

The most promising subject in the curriculum can be turned into a formal and intellectually stagnant drill if it is presented by a teacher who has no breadth of outlook and no desire to teach pupils how to generalize experiences.

It is not far from the truth to assert that any subject taught with a view to training pupils in methods of generalization is highly useful as a source of mental training and that any subject which emphasizes particular items of knowledge and does not stimulate generalization is educationally barren.[1]

In discussions of aims and values of mathematics it should always be kept in mind that the realization of aims depends upon the right kind of teaching.

In the following pages a list of general aims of the teaching of secondary-school mathematics is presented which has been compiled from a number of studies relating to objectives, from articles dealing with the subject, and from prefaces of the authors of several textbooks. No attempt has been made to determine the

[1] Charles H. Judd, *Psychology of Secondary Education*, pp. 422, 432.

relative values of the objectives or to list the detailed instructional materials which are to contribute to the objectives and which constitute the course of study. If the foregoing discussion, together with the list of objectives which follows, gives the teacher a general acquaintance with the objectives of mathematics and prevents him from following the textbook blindly and aimlessly, this chapter has served its purpose.

The objectives of the teaching of secondary-school mathematics.— The objectives which are given in the following pages have been classified as "powers," "appreciations," "understandings," "attitudes," "habits," and "skills." Since the list is intended to be only suggestive, it has been limited to secondary-school arithmetic, algebra, and geometry. Nothing will be said about the problems of finding the best methods of determining objectives or of making the course of study. They will be discussed fully in another volume.

The objectives of mathematics are as follows:

I. Power to think and to do:
   1. Powers which are to be developed in all subjects but to which mathematics should make definite contributions:
      a) To concentrate on a given task
      b) To carry a task to completion
      c) To use precise and simple language in oral and written form
      d) To systematize and classify given data
      e) To think logically and accurately through a problem
      f) To draw correct inferences
      g) To do neat and accurate work
      h) To do original thinking
      i) To analyze complex situations
      j) To recognize relationships
      k) To discover and plan solutions of problems
      l) To generalize conceptions
      m) To exercise constructive imagination
   2. Powers to be developed which relate particularly to mathematics in general:
      a) To express quantitative facts in mathematical language
      b) To recognize, analyze, and represent mathematical relationships
      c) To discover and formulate mathematical laws
      d) To use graphical representation
      e) To use mathematical methods

f) To use symbolic notation
g) To pre-estimate results
h) To use the mathematical concepts, laws, and processes in problems and in quantitative situations
i) To determine the degree of accuracy possible with measured data
j) To analyze quantitative situations into simple factors
k) To do functional thinking
l) To generalize mathematical concepts and processes

3. Specific mathematical powers. These objectives are more specific than the general mathematical objectives in the sense that they are closely related to the various mathematical subjects.

a) Arithmetical power:
(1) To read and understand numbers
(2) To use accurately and with a moderate degree of speed the fundamental processes with whole numbers, and common and decimal fractions
(3) To solve everyday numerical problems
(4) To use arithmetic in solving business problems
(5) To use short cuts where they save time and work
(6) To determine the number of figures to be retained in a computation
(7) To estimate the approximate result of a problem
(8) To extract the square root of a number
(9) To use the metric system of measuring

The following are not strictly mathematical objectives, but are traditionally assigned to certain mathematical courses:
(10) To keep family accounts
(11) To check bills
(12) To solve problems in percentage, discount, commission, interest, profit, and overhead
(13) To make personal and family budgets

b) Algebraic power:
(1) To understand reading in current literature containing algebraic statements and facts
(2) To use algebraic symbolism
(3) To understand and use the language of algebra
(4) To perform the fundamental operations of algebra with positive and negative numbers
(5) To perform the fundamental operations with monomials and polynomials

(6) To perform the fundamental operations with fractions
(7) To combine and decompose polynomials, e.g., to factor polynomials of the form ax^2+bx+c, including the special case a^2-b^2, and to factor polynomials having a factor common to all terms
(8) To understand and use formulas
(9) To solve formulas for a given letter
(10) To evaluate formulas
(11) To translate verbal statements into formulas
(12) To represent formulas graphically
(13) To express quantitative relationships algebraically
(14) To analyze problem situations
(15) To analyze relationships of quantity
(16) To solve problems by algebraic methods
(17) To solve linear, integral, and fractional equations
(18) To solve simultaneous linear equations algebraically and graphically
(19) To solve quadratic equations algebraically and graphically

c) Geometric power:
(1) To recognize geometric forms in nature, art, and industry
(2) To understand and make two- and three-dimensional drawings
(3) To use correct speech in discussions and proofs of geometric facts and principles
(4) To do neat and accurate work with geometric drawings
(5) To measure directly and indirectly
(6) To exercise spatial imagination
(7) To use symbolic notation in proofs and diagrams
(8) To analyze geometric situations
(9) To discover new geometric facts
(10) To recognize geometric relationships
(11) To attack and solve problems of space
(12) To reason correctly
(13) To establish geometric facts by proof
(14) To use a variety of methods of proof
(15) To make the fundamental geometric constructions

d) Power to be derived from the graphical method of representing numerical parts and relationships:
(1) To understand the meaning of graphical representation
(2) To interpret graphs
(3) To make graphs from given data

(4) To represent equations and formulas graphically
(5) To use the graphical methods of solving equations
(6) To study the properties of functions graphically

II. Appreciations:

1. Appreciations to be developed by mathematics in general:
   a) Of the contributions mathematics has made to the progress of civilization
   b) Of the influence of mathematics on the sciences, industries, and everyday life
   c) Of the relation of mathematics to the pupils' environment
   d) Of the value of mathematics to other school subjects
   e) Of the value of mathematics in vocations, business, industry, and architecture
   f) Of the mathematical modes of thinking
   g) Of the symbolism of mathematics
   h) Of the importance of neatness and accuracy
   i) Of dependence and relationships of facts in everyday life.
   j) Of the historical development of the mathematical subjects

2. Appreciations to be developed in connection with the study of arithmetic:
   a) Of the importance of mastery of the fundamentals
   b) Of the power of computation
   c) Of the usefulness of arithmetic in other mathematical subjects and in everyday life
   d) Of the degree of accuracy required in computations

   The following are non-mathematical objectives: Appreciation
   e) Of the need of care in making investments
   f) Of the advantages of placing money on interest
   g) Of the need for taxes
   h) Of the benefits derived from the community
   i) Of the duties toward the community
   j) Of the value of being thrifty

3. Appreciations to be acquired in geometry:
   a) Of the beauty of geometric forms in nature, art, and architecture
   b) Of symmetry, similarity, and regularity of figures
   c) Of space relationships
   d) Of the value of a logical reasoning
   e) Of the mathematical proof

4. Appreciations developed in algebra:
   a) Of the value of algebraic notation and symbolism
   b) Of the value of algebra as a tool for solving problems
   c) Of the power of the formula and graph

III. Understandings:

1. Understandings to be developed in arithmetic:
   a) Of the number system
   b) Of the fundamental concepts of arithmetic

   The following are non-mathematical objectives: Understandings
   c) Of keeping accounts, making budgets
   d) Of business practices relating to profit and loss, overhead, costs, selling prices, bills, receipts, commission, discounts
   e) Of such essentials of banking as saving and checking accounts, interest, money orders, notes
   f) Of the essentials of good investments, such as stocks, bonds, mortgages, corporations
   g) Of the various types of insurance: fire insurance, life insurance, and others
2. Understandings in geometry:
   a) The fundamental concepts of geometry
   b) The units of measurement
   c) Space forms
   d) The mensuration formulas
   e) Important geometric theorems and axioms
   f) Relationships in space forms
3. Understandings in algebra:
   a) The fundamental concepts of algebra, such as the literal number, the negative number, the equation, the formula, and the exponent
   b) The laws of algebra with literal numbers
   c) The laws of algebra with positive and negative numbers

IV. Attitudes:

1. A degree of interest in mathematics which will encourage the pupil to continue in the study
2. Desire to read mathematical literature growing out of the pleasure to be derived from such reading
3. Desire to make precise statements
4. Desire to estimate in advance the solution of a problem
5. Desire for thoroughness and clearness

6. Willingness to concentrate on problems
7. Desire to analyze complex problem situations
8. Desire to carry tasks to completion
9. Desire to do neat written work
10. Desire to think logically
11. Attitude of inquiry
12. Desire to make discoveries
13. Desire to grow mentally, to improve former records
14. Desire to constantly improve one's methods, e.g., to prefer the algebraic method of solving problems to the less-effective arithmetic method
15. Desire to concentrate
16. Desire to understand
17. Desire to generalize
18. Desire to assume responsibility for an assigned task
19. Desire to pursue the study of mathematics
20. To consider the study of mathematics a pleasure
21. Self-confidence in studying mathematics, and in problem-solving

V. Habits and ideals:
1. Of carefulness
2. Of persistence
3. Of concentration
4. Of observation
5. Of participation
6. Of neatness in written work
7. Of accuracy
8. Of thoroughness
9. Of clearness
10. Of precision
11. Of interpreting results
12. Of using good language
13. Of checking results

VI. Skills:
1. In arithmetical computation
2. In algebraic processes
3. In using the instruments of geometry
4. In making graphs
5. In solving equations and formulas
6. In solving problems

Relation of the aims of the teaching of mathematics to teaching procedures.—The lists of objectives presented in the foregoing

pages are by no means complete, but they show that teachers must assume responsibilities beyond that of merely teaching detailed isolated subject matter. The assimilative materials that deserve to be included in a course must contribute to the attainment of definite objectives, primarily the unit objectives. Moreover, the study of mathematics must lead to definite general mathematical objectives, and it must also contribute to some of the general educational objectives.

With definitely chosen objectives before him the teacher faces the problem of selecting the most appropriate materials and methods of instruction. It is hardly necessary to add that his choice will also be influenced by the ability of the pupils and the type of school in which he teaches.

Waples and Stone present a method of solving of the problem by making a careful analysis of one unit.[1] They first selected the materials for teaching the unit on positive and negative numbers on the following bases:

1. By analysis of a number of textbooks
2. By carefully observing the pupils at work to record the pupils' activities in the study of the unit
3. By analysis of the items obtained from 1 and 2, to discover other items and activities
4. By examining other studies to make comparisons and to check omissions
5. By classifying the data under formal headings

The data were reclassified according to definite objectives of the unit. This gave a list of 30 outcomes of the unit. This list was carefully checked by a selected group of specialists, which led to 11 desired objectives of the unit:

1. To understand the meaning and practical use of signed numbers
2. To solve an equation in one unknown when positive or negative numbers are involved as coefficients
3. To add, subtract, multiply, and divide signed numbers with accuracy and a fair degree of speed
4. To extend the number system
5. To solve problems involving signed numbers
6. To evaluate expressions involving positive and negative numbers
7. To add numbers graphically and by rule

[1] Douglas Waples and Charles A. Stone, *The Teaching Unit*, chap. ii.

8. To understand the connection between directed numbers and the coordinate system
9. To solve equations containing two unknowns involving positive and negative numbers
10. To recognize the use of directed numbers in connection with formulas, thus extending their use
11. To distinguish between plus or minus signs as signs of quality and signs of operation

These findings are indeed interesting. A glance at the list makes it apparent that the group of specialists feels that the objectives of the unit on positive and negative members cannot be attained until the pupil has studied the operations, equations, formulas, and problems in which signed numbers occur.

The investigation renders a real educational service by defining a technique of determining the assimilative materials and objectives for a particular unit. The procedure can be easily modified and adapted to any other unit. It is capable of such simplifications as to be of practical use to teachers.

Summary.—The chapter has emphasized the importance of a clear understanding of the objectives of mathematics in teaching the subject. Objectives have been classified as: general educational objectives, general mathematical objectives, subject objectives, and unit objectives. Before teaching a unit, the instructor must decide definitely on the contributions which a unit is to make to these objectives.

Several systems of objectives of secondary education have been presented, namely, those of Spencer, Bobbitt, the Commission on the Reorganization of Secondary Education, and the Subcommittee on Junior High School Mathematics.

The chapter also discusses the mathematical objectives presented by the National Committee on Mathematical Requirements, Schultze, Young, and Schorling.

Finally, a list of objectives has been compiled which is suitable for junior high school work in mathematics. The objectives in the list have been classified as powers, appreciations, understandings, attitudes, habits, and skills. Each class has been subdivided into general mathematical objectives and subject objectives.

The chapter closes with a discussion of an attempt by Waples and Stone to develop a technique for determining unit objectives.

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CHAPTER IX

TENDENCIES IN SELECTING AND ORGANIZING INSTRUCTIONAL MATERIALS

Dissatisfaction with the mathematical curriculum.—Mathematics is the oldest of all the sciences. It was one of the first subjects to be admitted to the secondary-school curriculum, and it has passed through a period of development and adjustment extending over centuries. Nevertheless, courses in secondary-school mathematics are still far from satisfactory. Evidence of this assertion is easily found in the numerous criticisms of arithmetic, algebra, and geometry which come from parents of pupils, business men, and instructors of college mathematics, and in the results of certain investigations which have revealed that an abnormally large number of pupils fail to profit sufficiently from the study of mathematics to make passing grades. Indeed, some educators have seriously questioned the value of the subject. They claim that, comparatively, the subject has stood still while the whole system of school organization was progressing, and that from the standpoint of a modern curriculum the content of the mathematical courses is poorly selected and arranged. It is generally agreed that a reorganization is needed, but the attempts to find a better organization have encountered the most baffling problems.

The recommendations for curriculum construction are varied, and often depend entirely on the point of view of the curriculum-maker. For example, the difficulties which pupils experience in the study of mathematics suggest a reconstruction for the purpose of eliminating, or at least minimizing, such difficulties. A totally different type of reconstruction may grow out of attempts to attain the general educational objectives. However, it is altogether possible that upon closer investigation the seemingly conflicting tendencies which develop out of differences in point of view may be brought into harmony with each other.

Curriculum tendencies growing out of problems relating to the teaching of traditional algebra and geometry.—Let us examine some curriculum tendencies arising from problems related to the arithmetic which is used in the study of high-school mathematics. High-school teachers of algebra have always complained that elementary-school graduates are not sufficiently well grounded in the fundamental arithmetical operations, that they lack the skill in problem-solving to be expected of pupils entering the high school, and that they do not understand the most commonly used terms, such as "percentage," "interest," "discount," etc. However strange this may seem in view of the fact that all fundamental processes have been taught before the pupil reaches the seventh grade, this particular problem is leading to a thorough reconstruction of the arithmetical curriculum of the seventh and eighth grades. A new arithmetic for these grades is rapidly developing.

One of the methods used in solving the problem is to ascertain by careful analysis exactly what knowledge of arithmetic is essential to a successful study of algebra and geometry. From such studies two significant conclusions have been drawn: First, the seventh- and eighth-grade curriculum must put more emphasis on *simple* arithmetic (i.e., on small numbers and simple operations) and on greater accuracy in computational work. Second, after the sixth grade ample opportunity must be provided for more constant use of arithmetic in a large variety of problem situations. These investigations also show that pupils do not really attain anything like arithmetical maturity before they reach the last two years of the high school. Hence, high-school teachers are learning to accept a reasonable share of responsibility for training in arithmetic and are providing systematic arithmetical instruction in the first two years of the high-school course. There is a growing tendency to provide more arithmetical training and experiences in connection with senior high school mathematics.

Certain pedagogical needs in the study of algebra are likewise introducing radical curriculum changes. Algebra is difficult because it is highly abstract and mechanical. Moreover, progress in the subject is too rapid, and not enough time is allowed to provide the experiences necessary to acquire real understanding of such fundamental concepts as literal number, signed number, exponent, and

equation. This has led curriculum makers to advocate the introduction of a certain amount of algebra earlier than is customary. A new type of algebra is being developed which is of a practical character and which is given a place on the curriculum of the two grades preceding the year when algebra is to be studied as a science. This algebra differs from the traditional formal ninth-year algebra. Any attempt to move traditional algebra bodily into the lower grades is doomed to failure, for the subject has been found too difficult even for ninth-grade pupils. The solution of the problem is not in moving traditional algebra downward but in providing a different type of algebra for the seventh and eighth grades.

A similar tendency has developed out of the pedagogical needs of the teaching of high-school geometry. As in the case of algebra, the study of logical geometry does not offer sufficient concrete experiences to develop clear understandings of the meanings of the fundamental geometric terms. Furthermore, progress is too rapid for the development of an appreciation of the need and meaning of a logical proof. These matters call for more time and wider experiences. The attempts of finding the solutions of the two problems have led curriculum-makers to include in the curriculum of Grades VII and VIII a new type of geometry, commonly designated as "intuitive geometry," which prepares the learner gradually for the study of logical geometry.

Thus, the attempts to pave the way for the successful study of high-school mathematics have influenced the reorganization of the mathematical curriculum in the grades below the ninth.

Curriculum tendencies growing out of the needs of the pupil.— There are now available the results of several investigations which aim to determine the mathematical needs of pupils. Some of them relate to the mathematical needs of the pupils in school subjects other than mathematics, such as sewing, sheet-metal courses, woodwork, printing, geography, electricity, and others. The interesting thing about these results is that the needs of the various subjects are actually not different from those of the mathematical courses. The striking feature of the arithmetic used in other courses is its simplicity as to operations, size of numbers, units of measure, and vocabulary. The arithmetical material traditionally taught in grades VII and VIII is far greater than is actually used, but the

investigations show clearly that accuracy in all operations cannot be overemphasized.

The findings do not show any need for demonstrative geometry; but they reveal many geometric concepts and principles that the pupil should know, and a variety of activities in reading, measuring, and drawing which presuppose clear understanding of these concepts and principles.

Finally, it is found that comparatively little algebra is used but that there are numerous instances in which some simple algebra could be used advantageously.

The investigations relating to the social needs of the everyday life of pupils and adults show similar results. The arithmetical fractions which occur generally have small denominators. A good knowledge of percentage is required. The fundamental operations with integers and fractions are of a simple type. It is further found that the social needs of adults call for acquaintance with the meaning of such terms as "insurance," "banking," "investments," "interest," "discount," "commission," etc. Topics dealing with these concepts should therefore be included in the curriculum. However, it should be clear that the purpose in offering such topics is not primarily training in arithmetical computation but understanding of the meaning of the terms in order that the pupil may be able to follow discussions relating to them when they are carried on at home or elsewhere. Adults use practically no algebra, probably because they do not understand the subject sufficiently well to prefer the algebraic method of problem-solving to the more clumsy arithmetical method.

The foregoing discussion has shown (1) that in constructing a curriculum in mathematics for any given age level, we should take into consideration the needs of the upper courses in mathematics, the needs of the pupil in subjects other than mathematics, and the needs of the pupil in everyday life; and (2) that for the first two or three years of the secondary school, all of these needs suggest very much the same type of material.

Furthermore, our experience in curriculum-making is not different from that of other countries. In practically all leading countries of Europe, systematic geometry of the intuitive type is begun one or two years before the seventh school year. In all of them, the

study of algebra in a simple form is taken up earlier than in American schools. Arithmetic, algebra, and geometry are taught simultaneously for several years; and the change to logical geometry and scientific algebra is very gradual.

The historical argument for the reorganization of mathematics.—The tendency to introduce certain phases of algebra and geometry before the ninth grade is strengthened by the theory that the individual should learn mathematics very much as the race has learned it, and that he should accordingly pass through the various stages of the historical development. The outstanding factors in this development which are of importance in the problem of reconstructing the mathematical curriculums are briefly as follows:

It took the race thousands of years to reach the stage at which the study of mathematics now begins. The origin of mathematics is counting, measuring, and drawing. Thus, arithmetic and geometry are the foundations of mathematics, and therefore geometry should have a place in the early curriculum as well as arithmetic. Furthermore, nearly all the early peoples developed some simple geometric forms. They made designs which they used for the purposes of decoration on clothing, draperies, rugs, and buildings. Progress in the development of this type of geometry was gradual and improvement steady. Both the Egyptian and Greek civilizations, which have contributed so much to geometry, passed through the early stages, proceeding by steps to more advanced forms until they reached the highest stage of beauty and art.

However, the Egyptians developed still another type of geometry. As early as 1850 B.C. they are known to have carried on extensive projects of irrigation, involving measuring, leveling, and surveying. Gradually they acquired a considerable knowledge of geometric relationships which enabled them to design and build their temples and pyramids. Thus, being of a practical mind, they developed a useful, practical type of geometry, but they were unable to advance farther in the subject.

Finally, the Greek mind began to apply to geometry the processes of pure reasoning; and within three hundred years the Greeks developed a third type of geometry, which we may call "logical geometry" because it organized all the facts and principles of geometry into a logical sequence. Thus, the essentials of the course

in high-school geometry were developed nearly three hundred years before the time of Christ. The later developments of the subject have been assigned to the field of advanced mathematics.

The beginning of algebra is as ancient as that of geometry. As long as seventeen hundred years before the time of Christ, Egyptian mathematicians were able to solve problems leading to equations in an unknown quantity. They were unable to proceed very far, because, on account of a lack of proper and sufficiently simple symbolism, they were required to use clumsy verbal statements of equations. Indeed, the development of algebra occasionally stood completely still. Not until the thirteenth century did algebra pass from the first stage, i.e., the use of words, to the second stage, i.e., the use of abbreviations of words. Finally, symbolic notation, representing the third stage in the development of algebra, appeared at the close of the fifteenth century. Near the middle of the seventeenth century, the meanings of symbols of operation, parentheses, literal number, signed number, and exponent were fairly well established. Thus, high-school algebra in its present form is only a little over four hundred years old. The study of the tremendous difficulties which the race had to overcome in the development of this subject is exceedingly fruitful to the teacher of algebra and to the curriculum-maker.

The story of the development of algebra and geometry brings out several important facts. Logical geometry as taught in high schools is but one of several forms the development of the subject has assumed. In the past the other types have been seriously neglected in the curriculum. Furthermore, algebra, although developing much more slowly than geometry on account of certain almost insurmountable difficulties, has been assigned a place on the curriculum lower than that of geometry. From the historical point of view that order may be questioned. A modern curriculum should offer some systematic instruction in geometry, either before, or at least in connection with, algebra.

The location of algebra and geometry in the secondary-school curriculum is quite recent, and a study of the development of the curriculum of our secondary schools should reveal any reasons for the traditional order in which the two subjects are taught if they exist. The present curriculum had its beginning in America during

the first part of the eighteenth century. Courses in algebra and geometry were then offered as college subjects. They had been developed in English universities; and at first, all instruction in mathematics in American schools was given by men who either came from England or had gone to England or Scotland to study. Thus, our early colleges offered algebra and geometry to their students as logical sciences. Geometry was taught in the Senior year; and, because of its close relation to arithmetic, algebra was studied prior to geometry, i.e., in the Junior year. The courses were then gradually moved downward until they were given places in the Freshman and Sophomore years.

With the coming of the academies and high schools, the two subjects continued to move downward; and after the first half of the nineteenth century we find algebra generally in the first year of the high school, and geometry in the second, the sequence having been retained from the beginning.

It is thus seen that algebra and geometry were originally intended as studies for adults. They were in no sense organized to meet the needs, interests, and abilities of the pupils who are studying them today; and there is no justification in the development of the mathematical curriculum for the practice of presenting algebra and geometry in their advanced forms without previously offering systematic instruction in the preliminary forms in the lower grades.

Curriculum tendencies arising from the demand for psychological arrangement of materials.—There is another pedagogical argument for making this reform. In the past there has been a strong tendency, in organizing courses in mathematics, to arrange subject matter in a strictly logical sequence. According to this type of organization, it does not matter how simple a principle might be; it cannot be presented until the place is reached to which it has been assigned as a step in a logical arrangement. As a result, the study of much simple and valuable subject matter is unnecessarily long deferred. A striking example of this tendency is the course in trigonometry. Because it happens to make use of a few elementary geometric facts, it has been postponed until a whole year's work of geometry is finished. The fact is that algebra, geometry, and trigonometry all contain simple and complex principles; and the question is not whether one subject is harder than another, but whether it is feasi-

ble to bring together, at the beginning of secondary-school mathematics, the simple principles of all mathematical subjects, the complexity increasing as the pupil progresses. Such an arrangement would be more suitable to the beginner than an intensive study of one subject to the exclusion of all others.

Thus, it has been seen that from the standpoint of learning the tendency in high-school mathematics of introducing algebra and geometry in the seventh and eighth grades is fully justified. The junior high problem has strengthened this movement, but the problem is really one of curriculum construction of the seventh and eighth grades, regardless of any argument for or against the junior high school. How this organization is best brought about will be shown later. It is sufficient to point out that the organization of the curriculum of secondary-school mathematics must start with the work of the seventh grade.

At this place two questions have probably occurred to the reader. Previously, attention was called to the lack of arithmetical preparation of elementary-school graduates. Will not the introduction of algebraic and geometric materials reduce considerably the time now given to arithmetic? If so, how can the proposed reconstruction be expected to give *better* results? The cause of poor arithmetical preparation is not necessarily lack of time, for at present more than sufficient time is assigned to arithmetic; but it is the way the subject matter has been selected and presented. Several investigations have shown clearly that seventh- and eighth-grade pupils who have studied algebra and geometry in addition to arithmetic actually know their arithmetic better than those who have devoted the two full years exclusively to arithmetic. Such an investigation was carried on in several of the Chicago elementary schools and junior high schools. Although conditions were favorable to the elementary-school pupils who studied only arithmetic, the results were favorable to the junior high school group which divided the allotted time between arithmetic, algebra, and geometry.

The other question to be answered at this time is this: Is not the proposed organization justifiable only if high-school preparation is regarded as the major aim of the lower grades? This is not the case. We do not intend to overlook the fact that our most worthy pupils are those who continue in school work, but it has

been shown that the needs and interests of all pupils of these grades justify the same type of material that is required in the preparation for the further study of mathematics.

Recommendations of important committees.—It is correct to state that practically all important committees charged with the problem of reconstructing the mathematical curriculum have come practically to a common conclusion. The Committee of Ten in 1893 recommended instruction in algebraic symbols and in simple equations. It advocated that a child's geometrical training should begin as early as possible; that systematic instruction in concrete and experimental geometry should begin at the age of ten years, and that it should occupy one hour per week for at least three years. These recommendations followed closely the prevailing practice in Europe. It was not presumed that algebra and geometry in the early grades should appear as separate subjects but that they be taught in connection with arithmetic.

Exactly thirty years later, in 1923, the National Committee on Mathematical Requirements says in its report:

> In recent years there has developed among many progressive teachers a very significant movement away from the older rigid division in "subjects" such as arithmetic, algebra and geometry, each of which is "completed" before another is begun, and toward a rational breaking down of the barriers separating these subjects in the interest of an organization of subject matter that will offer a psychologically and pedagogically more effective approach to the study of mathematics. . . . . The advocates of this new method of organization base their claims on the obvious and important relations between arithmetic, algebra and geometry, which the student may grasp before he can gain any real insight into mathematical methods and which are inevitably obscured by a strict adherence to the conception of separate "subjects." . . . . The newer method enables the pupil to gain a broad view of the whole field of elementary mathematics early in his high school course. In view of the very large number of pupils who drop out of school at the end of the eighth or the ninth school year or who for other reasons then cease their study of mathematics, this fact offers a weighty advantage over the older type of organization under which pupils studied algebra alone during the ninth school year to the complete exclusion of all contact with geometry.[1]

Articulation of junior and senior high school mathematics.—One of the principal arguments in favor of the junior high school move-

[1] *Committee on Mathematical Requirements*, pp. 12, 13.

ment is that something must be done to eliminate the break which exists between the work of the elementary school and high school. It is therefore appropriate to show how the course just outlined contributes to the solution of that problem.

In intuitive geometry the meanings of fundamental terms of two- and three-dimensional geometry are clearly established through concrete experiences. A period of months is devoted to what is usually attempted in a few days or weeks in the traditional courses. Many geometric principles are being discovered. The way is thereby paved for a further study of the subject. Provision is also made for the transition from intuition to the formal logical proof. As the pupil develops reasoning powers, he is gradually given material on which to try his skill. As he continues, informal discussions and reasoning are more and more stressed. In time the pupil recognizes the need for more careful reasoning and sees the meaning of logical geometry. The intuitive and informal reasoning stages belong to the intuitive geometry period, but there is no abrupt change from one to the other.

Likewise, the proposed course offers the introductory experiences necessary to prepare the pupil for a successful study of algebra. Emphasis has been on training in thinking and on understanding rather than on mechanical performance, and ample time has been allowed to establish clear meanings of the fundamental algebraic concepts.

Present tendencies in curriculum construction.—One may ask to what extent the seventh- and eighth-year courses now conform with the proposed changes. Undoubtedly most of the elementary schools will for some time continue to lean toward the traditional curriculum, but many progressive individual teachers and schools are experimenting with the newer materials. In the junior high schools the situation is more promising than in the senior high schools. The newer textbooks show a steady improvement in the selection and organization of the geometric materials.

The algebraic content is, on the whole, the least satisfactory of the three subjects. The common criticism is that the new courses and textbooks still lack that definiteness and continuity so characteristic of the traditional courses. This is one reason why senior high school teachers are inclined to discount the progressive work

done in the lower grades and to continue to offer the traditional courses with but slight modifications. Radical reorganization of high-school courses may be expected only to the extent to which the reorganization of the lower courses becomes effective. However, we may briefly state some curriculum changes that are gaining momentum in senior high school mathematics.

1. There is a growing tendency toward correlation of the various mathematical subjects. This is due to the emphasis which functional thinking, graphical procedure, and the formula are receiving throughout the world; to the increasing practice of introducing more and more geometric materials to make concrete the fundamental concepts, principles, and processes of algebra; and to the wider use of algebraic notation and the equation in expressing relationships in geometry.

2. There is a demand for a psychological organization of mathematical materials in pedagogical units. Even in geometry, the sacred sequence of Euclid no longer exists, and pedagogical arrangement is now valued as highly as logical arrangement. There is no reason why the organization cannot be both logical and psychological.

3. Obsolete material is being eliminated. This involves a reduction of such topics as factoring, proportion, literal and radical equations, and exponents, and the entire omission of such topics as square root of polynomials, and the binomial theorem from first year algebra.

4. The problem of providing for individual differences is leading to a determination of the minimum to be learned by all pupils. On the other hand, the various courses continue to introduce supplementary materials for the more capable pupils.

5. High-school trigonometry will in time cease to be equivalent to identities and equations. Emphasis is on the practical phases of the subject, and the elementary part is gradually being moved downward. It is needless to say that trigonometry is a true correlation of arithmetic, algebra, and geometry.

In the light of the experiences of the last thirty years these changes are not expected to take place rapidly, on account of the well-known conservatism of high-school teachers; but even a superficial examination of courses of study and textbooks published dur-

ing a period of ten or more years shows unmistakably that definite progress is being made.

A plan of organizing the instructional materials which conforms to modern tendencies.—The first step in curriculum construction has revealed the need for a new type of material to be introduced in the early years of secondary-school mathematics. The second raises the question of arranging this material into an organization which will attain most effectively the aims and purposes of the study. This question in turn is closely related to those of determining the objectives of mathematical instruction and the learning units of each year's work. It is not intended to offer any lengthy discussion of both of these important topics, but only to indicate in broad outline the organization of the materials which constitute the mathematics required of all pupils.

It has been shown that the instructional materials to be included in the beginning courses in mathematics should be taken from the fields of arithmetic, algebra, and geometry. The arithmetic should be computational and should emphasize the practical and social values of mathematics. A great deal of arithmetic is to be offered, but it should not be the only subject taught. The algebra should be simple. It should be employed in situations in which it is more helpful than arithmetic. Sufficient experiences should be provided with problems and formulas to develop clear conceptions of the fundamental notions of the subject. Geometry should include form study leading to clear understanding of the most fundamental geometric concepts and principles.

Because of its concreteness, geometry might well be made the basis of the mathematical curriculum of the seventh and eighth grades. If taught as practical measurement, geometry introduces a large variety of applications of arithmetic, thereby establishing a correlation between the two subjects. At the same time the use of literal notation should be encouraged. This is the beginning of algebra. Gradually algebraic principles are developed whenever they are needed in the manipulation of numerical or geometric materials. Emphasis is to be on simplicity, clearness, and on understanding rather than on blind submission to rules. Thus, each of the three subjects illustrates and illuminates the other two. Especially does the manipulation of algebraic formulas offer excellent training in

arithmetical computation. The correlation between the various mathematical subjects is most natural in the beginning stage.

The instructional materials may be best presented on three levels dealing respectively with lengths, areas, and volumes. In the first the pupil passes systematically through the study of lines, angles, triangles, polygons, and circles. The activities are those of observing, measuring, comparing, drawing, constructing with ruler and compasses, and discovering facts and relationships. In all of this work a small amount of algebra is needed, and there is demand for algebraic expressions or equations of the first degree but not higher than the first. Thus, to unknown length corresponds the unknown literal number a. The circumference of the circle is $3.14\ d$. The sum of two unmeasured segments is the binomial $a+b$. The perimeter of a triangle is $a+b+c$; and of a polygon $a+b+c+d+\ldots\ldots$ Geometric problems furnish ample practice in solving such simple equations as $3.14d=20$, $3x+x+5x=180$, etc. Practice in arithmetic is derived from evaluating algebraic expressions and from solving equations and problems. Graphical work is introduced as a means of representing number facts by straight lines and circles. Progress is slow but steady, and enough time must be allowed for acquiring clear understanding of the fundamental concepts of mathematics. As much as three-fourths of the time assigned to the mathematics in the seventh grade may be spent in studying the materials of the first level.

The second level of intuitive geometry is concerned with the measurement of surfaces. It develops the formulas for finding the areas of common plane figures and discovers many of their properties. For example, the theorem of Pythagoras is taught. Simultaneously, such second degree expressions as bh, $\frac{1}{2}bh$, a^2, $\frac{a^2}{4}\sqrt{3}$, $\frac{1}{2}h(a+b)$, and πr^2 are introduced. There is no need for algebraic expressions of degree higher than the second, and none are presented. The study of arithmetic is extended to include square root.

The third level deals with the study of volumes. In connection with three-dimensional figures it brings in such algebraic expressions of the third degree as x^3, $\pi r^2 h$, abc, $\frac{1}{3}\pi r^2 h$, and $\frac{4}{3}\pi r^3$. In this, as in the other levels, there is no scarcity of problem material which offers training in the fundamental operations of arithmetic.

Attention should be called to the fact that literal numbers, but not signed numbers, occur in the study of intuitive geometry. An additional unit is therefore to be organized in which the meaning of positive and negative numbers is taught. Furthermore, a unit must be provided to include such community arithmetic as does not readily correlate with algebra and geometry.

This completes the curriculum of Grades VII and VIII. The course as outlined comprises the materials needed by pupils in everyday life, in other school subjects, in vocations, and in preparation for further study of mathematics as far as they can be presented at this age level. It constitutes the minimum to be required of the pupils who continue the study of mathematics, as well as of those who do not.

Summary.—This chapter aims to give the reader a broad view of modern tendencies in the reconstruction of the mathematical curriculum, and of the principles on which the tendencies are based.

Some of them grow out of problems relating to the laws of learning, and aim to prepare the student better for the mathematics of the senior high school. They are as follows:

1. The secondary schools are to continue to offer an abundance of experiences with arithmetical computation. Emphasis must be more on accuracy than on speed.
2. Some algebra is to be taught previous to the time when algebra is studied as a science.
3. Some geometry is to be taught in Grades VII and VIII.

These tendencies are dictated by the demands of the upper courses, but they also conform closely to the historical development of mathematics.

There is no reason why some geometry should not be taught earlier than algebra. The traditional order, according to which algebra comes first, has no justification in the development of the secondary curriculum.

The mathematics of each grade are being selected and organized to provide for the mathematical needs of the pupil in his other school subjects and in everyday life. It has been found that these various needs are filled by practically the same type of mathematical material.

The recommendations of important committees lean strongly in the direction of the foregoing tendencies.

The chapter suggests briefly a plan for organizing a mathematical curriculum which conforms to modern tendencies.

Attention is called to the need and method of articulating junior and senior high school mathematics.

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CHAPTER X

THE CO-OPERATIVE FUNCTIONS OF THE TEACHER OF MATHEMATICS

The scope of the teachers' activities.—In the foregoing chapters emphasis has been placed on discussions of teachers' activities that relate to instruction and classroom management, because the most important activities which the teacher must learn to perform successfully are those that relate to his classwork. The importance of improving instruction cannot be overemphasized. The beginning teacher just taking up work in a new school will probably have to spend most of his spare time of the first year thinking of ways of improving his classroom activities, adjusting himself to the life of the school, and acquainting himself with its traditions and policies.

However, it is not to be inferred that the teachers' responsibilities toward the school are over when the last class of the day has been taught. A teacher might be successful in the performance of his classroom duties but contribute little or nothing to the general welfare of the school. Efficient instruction is essential, but it is the "minimum" essential to be expected of the teacher. To be sure, most of his time will always have to be occupied with giving instruction, managing classes, and making preparations for both. Sooner or later, however, his influence must be felt outside of the domain of his classroom if he expects to remain a permanent member of the school and the community. He must make a definite contribution to the effectiveness of the school. For this reason this last chapter is being devoted to discussions of teachers' duties other than those of teaching and managing classes.

Departmental duties of the teacher of mathematics.—Each teacher must accept his share of the responsibility of making the work of the department as effective as possible. This calls for cordial relationships among the teachers and for full co-operation in formulating and supporting departmental policies. Without this co-operation the department will lack unity, and its efficiency will be greatly

impaired. The following are departmental duties which the teacher of mathematics must perform:

1. Attending and participating in all departmental meetings.
2. Participating in committee work. The following illustrates the nature of some of the work which might be done by committees:
   a) Reorganizing the course of study
   b) Selecting textbooks
   c) Setting up objectives of the various courses
   d) Selecting and recommending equipment
   e) Bringing to the attention of the teachers new and valuable books relating to the teaching of mathematics, and interesting articles which have appeared in the current mathematical literature
   f) Making and marking departmental tests
   g) Planning departmental exhibits, contests, or assembly programs
3. Conducting a remedial class after school.
4. Advising pupils who are engaged in supplementary projects.
5. Sponsoring the mathematics club of the school.
6. Preparing pupils for examinations other than those given by the department.
7. Administering standardized tests and scoring the papers.
8. Conferring and co-operating with the teachers offering the same courses in regard to special difficulties, the amount of time to be allowed for the various units, and other matters of common interest.

It is not expected that every teacher should engage in all of the foregoing activities. In general, each teacher should find the type of work he feels he is best prepared to do, and then assume the responsibility for it. Thus, one teacher may prefer to take care of the department's remedial work, another would rather be the adviser for the club, a third chooses to arrange for and direct the contests, etc. The burden of carrying these functions should be equally divided among the teachers. Not more than two hours a week per teacher should be required to perform the extra-class activities of the department.

Activities contributing to the efficiency of the school.—The most excellent policies of an administration of a school will fail if the teachers of the school do not give them the hearty co-operation which they deserve. The administration is responsible for the development of policies that offer the pupils the best possible opportunities for growth and development. The teacher accepting a posi-

tion in a school tacitly agrees to carry out the school's policy. Hence, if the administration chooses to employ a particular teaching technique, or to use a new marking system, or to try a certain method of grouping pupils, it deserves, and depends on, the most hearty support of the staff. The teacher who does not recognize this responsibility and fails to assume it does not have a clear conception of his work. Without it the school will accomplish little. The teacher must therefore develop a professional attitude, show a genuine educational interest, and take an active part in matters that are of importance to the school as a whole. The work of the school is a co-operative undertaking. The teacher cannot expect to continue indefinitely to receive the assistance, guidance, advice, and backing of the administration unless he is willing to make real contributions to some of the work to be done by the administration. The following activities offer opportunities to teachers to become influential factors in the life and work of the school:

1. Attending and participating in all faculty meetings called by the principal.
2. Working on committees appointed by the principal.
3. Entertaining cordial relationships with the teachers of all departments.
4. Co-operating with other departments, e.g., with the English department in improving the oral and written English of the pupils.
5. Taking part in such pupil activities as excursions, social events, debates, and entertainments.
6. Being faculty sponsor of a class, club, school paper, school annual, or school team.
7. Supporting the policies of the school librarian and securing the departmental co-operation for the school library.
8. Interpreting and carrying out the administrative policy of the school.
9. Co-operating in matters concerning the school plant, such as caring for equipment and eliminating waste.
10. Assuming responsibility for maintaining good order and school discipline in corridors, in passing of classes, and on the school grounds.

Extra-curriculum activities are of major importance to the all-round development of the pupil. The teachers have to assume the supervision of these activities. Not much time needs to be required for this work. In most cases one hour a week per teacher should be

sufficient. It should be understood that as far as possible the teacher should be given the choice of the particular activity which he feels qualified to direct or supervise.

Administrative duties of the teacher of mathematics.—In order to carry on its work effectively, the school administration must delegate to the teachers many tasks that are of an administrative character. The teacher must therefore acquaint himself with the practice of the school in respect to such duties, familiarize himself thoroughly with the type of work that he might be expected to do, and give the matter prompt and full co-operation. The following are illustrations of administrative duties that may be delegated to teachers:

1. Collecting such data and keeping such records about pupils and classes as are needed by the administration, e.g., records of attendance, absences, and tardinesses, or of particular deficiencies of individual pupils.

2. Making reports to the office on matters on which the administration needs to be informed. The following illustrate the nature of such reports: habitually tardy pupils; irregular attendance; delinquency; deficiency in important phases of school work, such as oral and written English; physical defects; pupils who show signs of illness.

3. Responding promptly to such occasional requests by the office as the following: making out special reports on individual pupils; giving tests that are part of an administrative program; making requisition for supplies or library materials; and supervising a part of the building when the administration officers are absent.

4. Outlining supplies, equipment, textbooks, etc., required by the principal as a basis for annual budget-making.

Duties of the teacher toward the community.—Often the community has needs which demand the co-operation of the teachers. It is most desirable to give this co-operation, for it will establish a friendly spirit of relationship between the school and the community. Ultimately this will be of benefit to the school, because it needs the support of the community in carrying out its educational policies. The following activities are typical:

1. Meeting socially with various groups of the community, e.g., with occupational groups, social groups, parents' association, and graduates of the school.

2. Giving advice and assistance to such groups, and promoting their interests. The teachers can do much to provide for the local needs of the

community by working on or with committees appointed by business men's clubs, farmers' organizations, labor organizations, the church, the Y.M.C.A., or the rotary club.

3. Collecting local materials for the courses offered in the school. Relationships with the foregoing groups will enable the teacher to find much material that is valuable in instruction. The shops, engineering offices, life insurance companies, banks, and business houses have an abundance of valuable instructional material that should be discovered and taught to the pupils. Exhibits or other publicity of this type of work done by the schools will convince the public that the schools are trying to serve the community. They will raise the prestige of the school in the community.

4. Taking an active interest in libraries, museums, and in literary or musical societies of the community.

5. Selecting speakers from the leaders of the community and engaging them to address classes, clubs, and assemblies.

6. Showing courtesy to school visitors.

Activities that develop professional growth of the teacher.—The teacher must not allow his professional growth and development to be stunted. He must keep in contact with modern tendencies, and progressive movements and methods. Furthermore, he must cultivate a research attitude which will make him alert in interpreting the results of tests or in analyzing the errors and difficulties of pupils. The following activities contribute to professional and personal advancement:

1. Reading professional literature. This includes general educational books and magazines, mathematical journals, and books on the teaching of mathematics. The teacher should not only know what is going on in his special field but should keep in contact with broad educational tendencies.

2. Taking courses. Correspondence schools, summer schools, and extension classes offer ample opportunity for taking professional courses. An occasional correspondence course, or a summer spent in studying at a university, will contribute greatly to the professional development of the teacher.

3. Taking active part in teachers organizations and institutes. Organizations are usually the agencies for starting new movements and developing new ideas. It is, therefore, strongly to be advised that teachers identify themselves with progressive organizations and take active part on the programs either as officers of the association or as members of committees or as speakers.

4. Visiting classes and observing other teachers. From such visits

much is to be learned about good teaching. Often a visit is valuable because the visitor sees how not to teach.

5. Undertaking the solution of a research problem. There is much to be determined about the mental processes of the pupils, the causes of errors and difficulties, and the superiority or inferiority of methods of instruction. No one is in a better position to secure evidence regarding such problems than the teacher who daily observes the pupil, who is in a position to collect an abundance of test material that may be analyzed, and who may try out various teaching procedures by experimenting with them in classes of comparable ability, under controlled test conditions. Many examples of research carried on by teachers are found in the foregoing chapters. Thus, references were made to the study of the effectiveness of the lecture method (p. 3), to the use of tests as a means of making diagnoses (p. 170), to the discovery of typical errors made by pupils (pp. 171–82), and to many other problems whose solutions may be utilized by the teacher.

Summary.—In the foregoing pages it has been shown that a school is a co-operative institution. Each teacher is a specialist in his own field, but his work extends beyond the classroom. He owes responsibilities to his department, to the school, and to the community. His reputation as a teacher will depend as much on what he does for the school as on his efficiency in the classroom. Moreover, no matter how busy he might be, he must not neglect his professional growth and development. By carrying on research work, he will extend his influence and worth beyond the school and the community, since he will be able to render valuable services to the teaching profession.

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